

Time Value of Money

CHAPTER

28

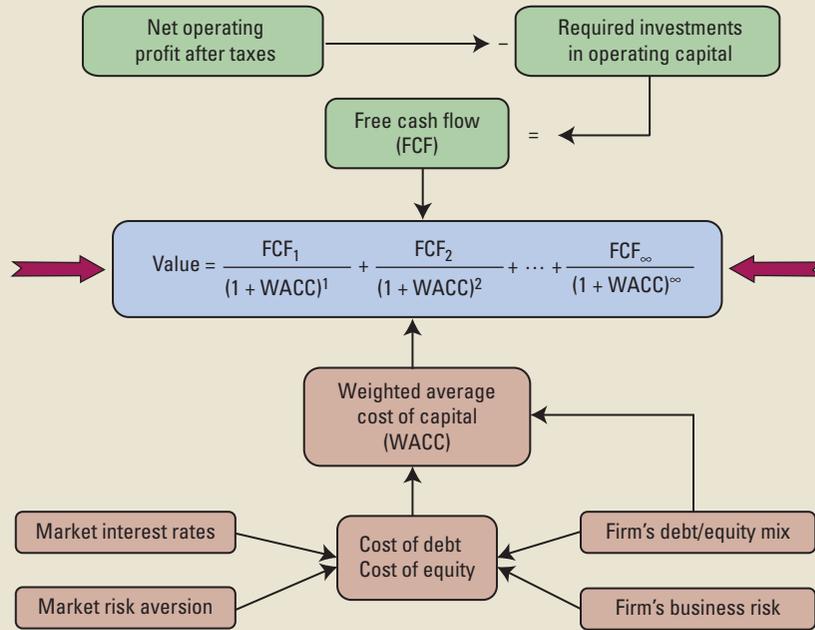
In Chapter 1, we saw that the primary objective of financial management is to maximize the intrinsic value of a firm's stock. We also saw that stock values depend on the timing of the cash flows investors expect from an investment—a dollar expected sooner is worth more than a dollar expected further in the future. Therefore, it is essential for financial managers to understand the time value of money and its impact on stock prices. In this chapter, we will explain exactly how the timing of cash flows affects asset values and rates of return.

The principles of time value analysis have many applications, including retirement planning, loan payment schedules, and decisions to invest (or not) in new equipment. *In fact, of all the concepts used in finance, none is more important than the **time value of money (TVM)**, also called **discounted cash flow (DCF) analysis**.* Time value concepts are used throughout the remainder of the book, so it is vital that you understand the material in this chapter and be able to work the chapter's problems before you move on to other topics.

CORPORATE VALUATION AND THE TIME VALUE OF MONEY

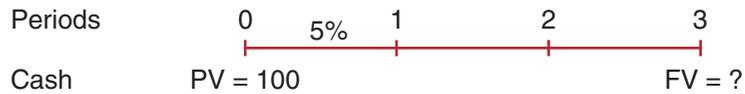
In Chapter 1, we explained: (1) that managers should strive to make their firms more valuable, and (2) that the value of a firm is determined by the size, timing, and risk of its free cash flows (FCF). Recall from Chapter 6 that free cash flows are the cash flows available for distribution to all of a firm's investors (stockholders and creditors). We explain how to calculate the weighted average cost of capital (WACC) in Chapter 11,

but it is enough for now to think of the WACC as the average rate of return required by all of the firm's investors. The intrinsic value of a company is given by the following diagram. Note that central to this value is discounting the free cash flows at the WACC in order to find the value of the firm. This discounting is one aspect of the time value of money. We discuss time value of money techniques in this chapter.



28-1 Time Lines

The first step in a time value analysis is to set up a **time line** to help you visualize what's happening in the particular problem. To illustrate, consider the following diagram, where PV represents \$100 that is in a bank account today and FV is the value that will be in the account at some future time (3 years from now in this example):



WEB

The textbook's Web site contains an *Excel* file that will guide you through the chapter's calculations. The file for this chapter is **Ch28 Tool Kit.xls**, and we encourage you to open the file and follow along as you read the chapter.

The intervals from 0 to 1, 1 to 2, and 2 to 3 are time periods such as years or months. Time 0 is today, and it is the beginning of Period 1; Time 1 is one period from today, and it is both the end of Period 1 and the beginning of Period 2; and so on. In our example, the periods are years, but they could also be quarters or months or even days. Note again that each tick mark corresponds to both the *end* of one period and the *beginning* of the next one. Thus, if the periods are years, the tick mark at Time 2 represents both the end of Year 2 and the beginning of Year 3.

Cash flows are shown directly below the tick marks, and the relevant interest rate is shown just above the time line. Unknown cash flows, which you are trying to find, are indicated by question marks. Here the interest rate is 5%; a single cash outflow, \$100, is invested at Time 0; and the Time-3 value is unknown and must be found. In this example, cash flows occur only at Times 0 and 3, with no flows

at Times 1 or 2. We will, of course, deal with situations where multiple cash flows occur. Note also that in our example the interest rate is constant for all 3 years. The interest rate is generally held constant, but if it varies, then in the diagram we show different rates for the different periods.

Time lines are especially important when you are first learning time value concepts, but even experts use them to analyze complex problems. Throughout the book, our procedure is to set up a time line to show what's happening, provide an equation that must be solved to find the answer, and then explain how to solve the equation with a regular calculator, a financial calculator, and a computer spreadsheet.

Do time lines deal only with years, or could other periods be used?

Set up a time line to illustrate the following situation: You currently have \$2,000 in a 3-year certificate of deposit (CD) that pays a guaranteed 4% annually. You want to know the value of the CD after 3 years.

SELF TEST

28-2 Future Values

A dollar in hand today is worth more than a dollar to be received in the future—if you had the dollar now, you could invest it, earn interest, and end up with more than one dollar in the future. The process of going forward, from **present values (PVs)** to **future values (FVs)**, is called **compounding**. To illustrate, refer back to our 3-year time line and assume that you have \$100 in a bank account that pays a guaranteed 5% interest each year. How much would you have at the end of Year 3? We first define some terms, and then we set up a time line and show how the future value is calculated.

PV = Present value, or beginning amount. In our example, $PV = \$100$.

FV_N = Future value, or ending amount, in the account after N periods. Whereas PV is the value *now*, or the *present value*, FV_N is the value N periods into the *future*, after interest earned has been added to the account.

CF_t = Cash flow. Cash flows can be positive or negative. For a borrower, the first cash flow is positive and the subsequent cash flows are negative, and the reverse holds for a lender. The cash flow for a particular period is often given a subscript, CF_t , where t is the period. Thus, $CF_0 = PV$ = the cash flow at Time 0, whereas CF_3 would be the cash flow at the end of Period 3. In this example, the cash flows occur *at the ends* of the periods, but in some problems they occur at the beginning.

I = Interest rate earned per year. (Sometimes a lowercase i is used.) Interest earned is based on the balance at the beginning of each year, and we assume that interest is paid at the end of the year. Here, $I = 5\%$ or, expressed as a decimal, 0.05. Throughout this chapter, we designate the interest rate as I (or I/YR , for interest rate per year) because that symbol is used on most financial calculators. Note, though, that in later chapters

we use the symbol “ r ” to denote the rate because r (for *rate* of return) is used more often in the finance literature. Also, in this chapter we generally assume that interest payments are guaranteed by the U.S. government and hence are riskless (i.e., certain). In later chapters, we will deal with risky investments, where the rate actually earned might be different from its expected level.

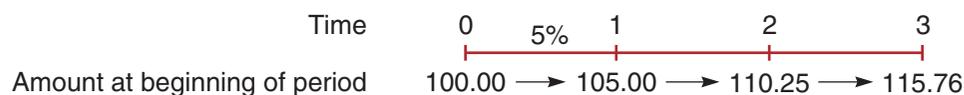
INT = Dollars of interest earned during the year = (Beginning amount) $\times I$. In our example, $INT = \$100(0.05) = \5 for Year 1, but it rises in subsequent years as the amount at the beginning of each year increases.

N = Number of periods involved in the analysis. In our example, $N = 3$. Sometimes the number of periods is designated with a lowercase n , so both N and n indicate number of periods.

We can use four different procedures to solve time value problems.¹ These methods are described next.

28-2a Step-by-Step Approach

The time line itself can be modified and used to find the FV of \$100 compounded for 3 years at 5%, as shown below:



We start with \$100 in the account, which is shown at $t = 0$. We then multiply the initial amount, and each succeeding beginning-of-year amount, by $(1 + I) = (1.05)$.

- You earn $\$100(0.05) = \5 of interest during the first year, so the amount at the end of Year 1 (or at $t = 1$) is:

$$\begin{aligned}
 FV_1 &= PV + INT \\
 &= PV + PV(I) \\
 &= PV(1 + I) \\
 &= \$100(1 + 0.05) = \$100(1.05) = \$105
 \end{aligned}$$

- We begin the second year with \$105, earn $0.05(\$105) = \5.25 on the now larger beginning-of-period amount, and end the year with \$110.25. Interest during

1. A fifth procedure is called the *tabular approach*, which uses tables that provide “interest factors”; this procedure was used before financial calculators and computers became available. Now, though, calculators and spreadsheets such as *Excel* are programmed to calculate the specific factor needed for a given problem, which is then used to find the FV. This is much more efficient than using the tables. Also, calculators and spreadsheets can handle fractional periods and fractional interest rates. For these reasons, tables are not used in business today; hence we do not discuss them in the text. However, because some professors cover the tables for pedagogical purposes, we discuss them in *Web Extension 28A*, on the textbook’s Web site.

Year 2 is \$5.25, and it is higher than the first year's interest, \$5, because we earned $\$5(0.05) = \0.25 interest on the first year's interest. This is called "compounding," and interest earned on interest is called "compound interest."

- This process continues, and because the beginning balance is higher in each successive year, the interest earned each year increases.
- The total interest earned, \$15.76, is reflected in the final balance, \$115.76.

The step-by-step approach is useful because it shows exactly what is happening. However, this approach is time-consuming, especially if the number of years is large and you are using a calculator rather than *Excel*, so streamlined procedures have been developed.

28-2b Formula Approach

In the step-by-step approach, we multiplied the amount at the beginning of each period by $(1 + I) = (1.05)$. Notice that the value at the end of Year 2 is:

$$\begin{aligned} FV_2 &= FV_1(1 + I) \\ &= PV(1 + I)(1 + I) \\ &= PV(1 + I)^2 \\ &= 100(1.05)^2 = \$110.25 \end{aligned}$$

If $N = 3$, then we multiply PV by $(1 + I)$ three different times, which is the same as multiplying the beginning amount by $(1 + I)^3$. This concept can be extended, and the result is this key equation:

$$FV_N = PV(1 + I)^N$$

(28-1)

We can apply Equation 28-1 to find the FV in our example:

$$FV_3 = \$100(1.05)^3 = \$115.76$$

Equation 28-1 can be used with any calculator, even a nonfinancial calculator that has an exponential function, making it easy to find FVs no matter how many years are involved.

28-2c Financial Calculators

Financial calculators were designed specifically to solve time value problems. First, note that financial calculators have five keys that correspond to the five variables in the basic time value equations. Equation 28-1 has only four variables, but shortly we will deal with situations where a fifth variable (a set of periodic additional payments) is involved. We show the inputs for our example above their keys in the following diagram, and the output, which is the FV, below its key. Because there are no periodic payments in this example, we enter 0 for PMT. We describe the keys in more detail below the diagram.

Inputs	3	5	-100	0	
	N	I/YR	PV	PMT	FV
Output					115.76

N = Number of periods = 3. Some calculators use n rather than N .

I/YR = Interest rate per period = 5. Some calculators use i or I rather than I/YR . Calculators are programmed to automatically convert the 5 to the decimal 0.05 before doing the arithmetic.

PV = Present value = 100. In our example we begin by making a deposit, which is an outflow of 100, so the PV is entered with a negative sign. On most calculators you must enter the 100, then press the $+/-$ key to switch from $+100$ to -100 . If you enter -100 directly, this will subtract 100 from the last number in the calculator, which will give you an incorrect answer unless the last number was zero.

PMT = Payment. This key is used if we have a series of equal, or constant, payments. Because there are no such payments in our current problem, we enter $PMT = 0$. We will use the PMT key later in this chapter.

FV = Future value. In our example, the calculator automatically shows the FV as a positive number because we entered the PV as a negative number. If we had entered the 100 as a positive number, then the FV would have been negative. Calculators automatically assume that either the PV or the FV must be negative.

As noted in our example, you first enter the four known values (N , I/YR , PV , and PMT) and then press the FV key to get the answer, $FV = 115.76$.

HINTS ON USING FINANCIAL CALCULATORS

When using a financial calculator, make sure it is set up as indicated below. Refer to your calculator manual or to our calculator tutorial on the textbook's Web site for information on setting up your calculator.

- **One payment per period.** Many calculators “come out of the box” assuming that 12 payments are made per year; that is, they assume monthly payments. However, in this book we generally deal with problems in which only one payment is made each year. *Therefore, you should set your calculator at one payment per year and leave it there. See our tutorial or your calculator manual if you need assistance.* We will show you how to solve problems with more than 1 payment per year in Section 28-15.
- **End mode.** With most contracts, payments are made at the *end* of each period. However, some contracts

call for payments at the *beginning* of each period. You can switch between “End Mode” and “Begin Mode” depending on the problem you are solving. *Because most of the problems in this book call for end-of-period payments, you should return your calculator to End Mode after you work a problem in which payments are made at the beginning of periods.*

- **Negative sign for outflows.** When first learning how to use financial calculators, students often forget that one cash flow must be negative. Mathematically, financial calculators solve a version of this equation:

$$PV(1 + I)^N + FV_N = 0$$

(28-2)

Notice that for reasonable values of I , either PV or FV_N must be negative, and the other one must be positive

to make the equation equal 0. This is reasonable because, in all realistic situations, one cash flow is an outflow (which should have a negative sign) and one is an inflow (which should have a positive sign). For example, if you make a deposit (which is an outflow, and hence should have a negative sign) then you will expect to make a later withdrawal (which is an inflow with a positive sign). *The bottom line is that one of your inputs for a cash flow must be negative and one must be positive. This generally means typing the outflow as a positive number and then pressing the +/- key to convert from + to - before hitting the enter key.*

- **Decimal places.** When doing arithmetic, calculators allow you to show from 0 to 11 decimal places on the display. When working with dollars, we generally specify two decimal places. When dealing with interest rates, we generally specify two places if the rate is expressed as a percentage, like 5.25%, but we specify four places if the rate is expressed as a decimal, like 0.0525.
- **Interest rates.** For arithmetic operations with a non-financial calculator, the rate 5.25% must be stated as a decimal, .0525. However, with a financial calculator you must enter 5.25, not .0525, because financial calculators are programmed to assume that rates are stated as percentages.

28-2d Spreadsheets

Spreadsheets are ideally suited for solving many financial problems, including those dealing with the time value of money.² Spreadsheets are obviously useful for calculations, but they can also be used like a word processor to create exhibits like our Figure 28-1, which includes text, drawings, and calculations. We use this figure to show that four methods can be used to find the FV of \$100 after 3 years at an interest rate of 5%. The time line on Rows 36 to 37 is useful for visualizing the problem, after which the spreadsheet calculates the required answer. Note that the letters across the top designate columns, the numbers down the left column designate rows, and the rows and columns jointly designate cells. Thus, cell C32 shows the amount of the investment, \$100, and it is given a minus sign because it is an outflow.

It is useful to put all inputs in a section of the spreadsheet designated “INPUTS.” In Figure 28-1, we put the inputs in the aqua-colored range of cells. Rather than enter fixed numbers into the model’s formulas, we enter the cell references for the inputs. This makes it easy to modify the problem by changing the inputs and then automatically use the new data in the calculations.

Time lines are important for solving finance problems because they help us visualize what’s happening. When we work a problem by hand we usually draw a time line, and when we work a problem with *Excel*, we set the model up as a time line. For example, in Figure 28-1 Rows 36 to 37 are indeed a time line. It’s easy to construct time lines with *Excel*, with each column designating a different period on the time line.

On Row 39, we use *Excel* to go through the step-by-step calculations, multiplying the beginning-of-year values by $(1 + I)$ to find the compounded value at the end of each period. Cell G39 shows the final result of the step-by-step approach.

2. The file *Ch28 Tool Kit.xls* on the book’s Web site does the calculations in the chapter using *Excel*. We highly recommend that you study the models in this *Tool Kit*. Doing so will give you practice with *Excel*, and that will help you tremendously in later courses, in the job market, and in the workplace. Also, going through the models will improve your understanding of financial concepts.

WEB

See *Ch28 Tool Kit.xls*
for all calculations.

FIGURE 28-1 Alternative Procedures for Calculating Future Values

	A	B	C	D	E	F	G
31	INPUTS:						
32	Investment = $CF_0 = PV =$		-\$100.00				
33	Interest rate = $I =$		5%				
34	No. of periods = $N =$		3				
35							
36	Time Line	Periods:	0	1	2	3	
37		Cash flow:	-\$100.00	0	0	0	FV = ?
38							
39	1. Step-by-Step: Multiply by (1 + I) each step			\$100.00 →	\$105.00 →	\$110.25 →	\$115.76
40							
41	2. Formula: $FV_N = PV(1+I)^N$			$FV_3 =$	$\\$100(1.05)^3$	=	\$115.76
42							
43		Inputs:	3	5	-100	0	
44	3. Financial Calculator:		N	I/YR	PV	PMT	FV
45		Output:					\$115.76
46							
47	4. Excel Spreadsheet:	FV function:	$FV_N =$	$=FV(I,N,0,PV)$			
48		Fixed inputs:	$FV_N =$	$=FV(0.05,3,0,-100) =$			\$115.76
49		Cell references:	$FV_N =$	$=FV(C33,C34,0,C32) =$			\$115.76
50	In the Excel formula, the terms are entered in this sequence: interest, periods, 0 to indicate no periodic cash flows, and then the PV. The data can be entered as fixed numbers or, better yet, as cell references.						
51							

Microsoft Excel® is a registered trademark of Microsoft Corporation. © 2014 Microsoft.

We illustrate the formula approach in Row 41, using *Excel* to solve Equation 28-1 to find the FV. Cell G41 shows the formula result, \$115.76. As it must, it equals the step-by-step result.

Rows 43 to 45 illustrate the financial calculator approach, which again produces the same answer, \$115.76.

The last section, in Rows 47 to 49, illustrates *Excel's* future value (FV) function. You can access the function wizard by clicking the f_x symbol in *Excel's* formula bar. Then select the category for Financial functions, and then the FV function, which is $=FV(I,N,0,PV)$, as shown in Cell E47.³ Cell E48 shows how the formula would look with numbers as inputs; the actual function itself is entered in Cell G48, but it shows up in the table as the answer, \$115.76. If you access the model and put the pointer on Cell G48, you will see the full formula. Finally, Cell E49 shows how the formula would look with cell references rather than fixed values as inputs, with the actual function again in Cell G49. We generally use cell references as function inputs because this makes it easy to change inputs and see how those changes

3. All functions begin with an equal sign. The third entry is zero in this example, which indicates that there are no periodic payments. Later in this chapter, we will use the FV function in situations where we have nonzero periodic payments. Also, for inputs we use our own notation, which is similar but not identical to *Excel's* notation.

affect the output. This is called “sensitivity analysis.” Many real-world financial applications use sensitivity analysis, so it is useful to form the habit of setting up an input data section and then using cell references rather than fixed numbers in the functions.

When entering interest rates in *Excel*, you can use either actual numbers or percentages, depending on how the cell is formatted. For example, we first formatted Cell C33 to Percentage, and then typed in 5, which showed up as 5%. However, *Excel* uses 0.05 for the arithmetic. Alternatively, we could have formatted C33 as a Number, in which case we would have typed “0.05.” If a cell is formatted to Number and you enter 5, then *Excel* would think you meant 500%. Thus, *Excel*'s procedure is different from the convention used in financial calculators.

Sometimes students are confused about the sign of the initial \$100. We used +\$100 in Rows 39 and 41 as the initial investment when calculating the future value using the step-by-step method and the future value formula, but we used -\$100 with a financial calculator and the spreadsheet function in Rows 43 and 48. When must you use a positive value and when must you use a negative value? The answer is that whenever you set up a time line and use either a financial calculator's time value functions or *Excel*'s time value functions, you must enter the signs that correspond to the “direction” of the cash flows. Cash flows that go out of your pocket (outflows) are negative, but cash flows that come into your pocket (inflows) are positive. In the case of the FV function in our example, if you invest \$100 (an outflow, and therefore negative) at Time 0 then the bank will make available to you \$115.76 (an inflow, and therefore positive) at Time 3. In essence, the FV function on a financial calculator or *Excel* answers the question, “If I invest this much now, how much will be available to me at a time in the future?” The investment is an outflow and negative, and the amount available to you is an inflow and positive. If you use algebraic formulas then you must keep track of whether the value is an outflow or an inflow yourself. When in doubt, refer back to a correctly constructed time line.

28-2e Comparing the Procedures

The first step in solving any time value problem is to understand what is happening and then to diagram it on a time line. Woody Allen said that 90% of success is just showing up. With time value problems, 90% of success is correctly setting up the time line.

After you diagram the problem on a time line, your next step is to pick one of the four approaches shown in Figure 28-1 to solve the problem. Any one approach may be used, but your choice will depend on the particular situation.

All business students should know Equation 28-1 by heart and should also know how to use a financial calculator. So, for simple problems such as finding the future value of a single payment, it is generally easiest and quickest to use either the formula approach or a financial calculator. However, for problems that involve several cash flows, the formula approach usually is time-consuming, so either the calculator or spreadsheet approach would generally be used. Calculators are portable and quick to set up, but if many calculations of the same type must be done, or if you want to see how changes in an input such as the interest rate affect the future value, then the spreadsheet approach is generally more efficient. If the problem has many irregular cash flows, or if you want to analyze alternative scenarios

using different cash flows or interest rates, then the spreadsheet approach definitely is the most efficient procedure.

Spreadsheets have two additional advantages over calculators. First, it is easier to check the inputs with a spreadsheet because they are visible; with a calculator the inputs are buried somewhere in the machine. Thus, you are less likely to make a mistake in a complex problem when you use the spreadsheet approach. Second, with a spreadsheet, you can make your analysis much more transparent than you can when using a calculator. This is not necessarily important when all you want is the answer, but if you need to present your calculations to others, like your boss, it helps to be able to show intermediate steps, which enables someone to go through your exhibit and see exactly what you did. Transparency is also important when you must go back, sometime later, and reconstruct what you did.

You should understand the various approaches well enough to make a rational choice, given the nature of the problem and the equipment you have available. In any event, you must understand the concepts behind the calculations, and you also must know how to set up time lines in order to work complex problems. This is true for stock and bond valuation, capital budgeting, lease analysis, and many other types of financial problems.

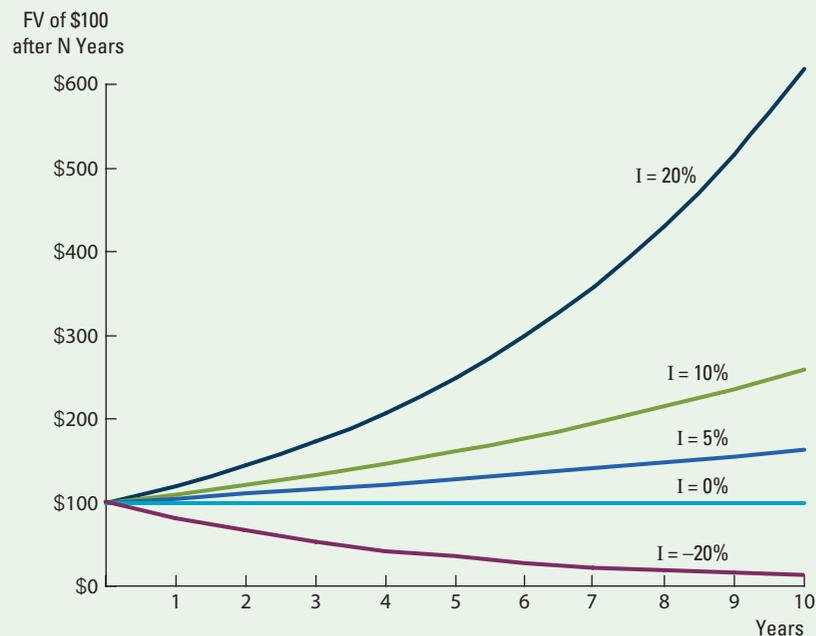
WEB

See *Ch28 Tool Kit.xls* for all calculations.

28-2f Graphic View of the Compounding Process

Figure 28-2 shows how a \$100 investment grows (or declines) over time at different interest rates. Interest rates are normally positive, but the “growth” concept is broad

FIGURE 28-2 Growth of \$100 at Various Interest Rates and Time Periods



enough to include negative rates. We developed the curves by solving Equation 28-1 with different values for N and I . The interest rate is a growth rate: If money is deposited and earns 5% per year, then your funds will grow by 5% per year. Note also that time value concepts can be applied to anything that grows—sales, population, earnings per share, or your future salary. Also, as noted before, the “growth rate” can be negative, as was sales growth for a number of auto companies in recent years.

28-2g Simple Interest versus Compound Interest

As explained earlier, when interest is earned on the interest earned in prior periods, we call it **compound interest**. If interest is earned only on the principal, we call it **simple interest**. The total interest earned with simple interest is equal to the principal multiplied by the interest rate times the number of periods: $PV(I)(N)$. The future value is equal to the principal plus the interest: $FV = PV + PV(I)(N)$. For example, suppose you deposit \$100 for 3 years and earn simple interest at an annual rate of 5%. Your balance at the end of 3 years would be:

$$\begin{aligned} FV &= PV + PV(I)(N) \\ &= \$100 + \$100(5\%)(3) \\ &= \$100 + \$15 = \$115 \end{aligned}$$

Notice that this is less than the \$115.76 we calculated earlier using compound interest. Most applications in finance are based on compound interest, but you should be aware that simple interest is still specified in some legal documents.

Explain why this statement is true: “A dollar in hand today is worth more than a dollar to be received next year, assuming interest rates are positive.”

What is compounding? What would the future value of \$100 be after 5 years at 10% *compound* interest? **(\$161.05)**

Suppose you currently have \$2,000 and plan to purchase a 3-year certificate of deposit (CD) that pays 4% interest, compounded annually. How much will you have when the CD matures? **(\$2,249.73)** How would your answer change if the interest rate were 5%, or 6%, or 20%? (*Hint:* With a calculator, enter $N = 3$, $I/YR = 4$, $PV = -2000$, and $PMT = 0$; then press FV to get 2,249.73. Then, enter $I/YR = 5$ to override the 4% and press FV again to get the second answer. In general, you can change one input at a time to see how the output changes.) **(\$2,315.25; \$2,382.03; \$3,456.00)**

A company’s sales in 2015 were \$100 million. If sales grow by 8% annually, what will they be 10 years later? **(\$215.89 million)** What would they be if they decline by 8% per year for 10 years? **(\$43.44 million)**

How much would \$1, growing at 5% per year, be worth after 100 years? **(\$131.50)** What would FV be if the growth rate were 10%? **(\$13,780.61)**

SELF TEST

IT'S A MATTER OF TRUST

One of our Founding Fathers, Benjamin Franklin, wanted to make a donation to the cities of Boston and Philadelphia, but he didn't want the cities to squander the money before it had grown enough to make a big impact. His solution was the "Methuselah Trust." When Franklin died in 1790, his will left £1,000, at that time about \$4,550, to Philadelphia and to Boston, but on the condition that it would be invested for 100 years, after which some of the proceeds were to be used for the public projects (primarily trade schools and water works) and the rest invested for another 100 years. Depending on interest rates, this strategy could generate quite a bit of money! For example, if half of his bequest, \$2,275, remained invested at 5% compound interest for the entire 200 years, the value in 1990 would be \$39.3 million! The ultimate payout, however, was only about \$7 million, because substantial amounts were eaten up by trustee fees, taxes, and legal battles. Franklin certainly would have been disappointed!

In 1936 an eccentric investor and New York lawyer, Jonathan Holden, decided to expand on Franklin's idea by donating a series of 500-year and 1,000-year trusts to

Hartwick College and several other recipients. By 2008 Hartwick College's trust had grown in value to about \$9 million; if invested at 5% for the remaining 928 years of its planned life, its value would grow to (\$9 million) $(1.05)^{928} = \$4.15 \times 10^{26}$. That is a lot of dollars by any measure! For example, that amount in million-dollar bills (if they existed by then) would paper the earth 10,000 times over, or laid end to end would reach the nearest star, Alpha Centauri, more than 1,000 times. In a move that surely would have disappointed Holden, Hartwick College was able to convert the trust into annual cash flows of about \$450,000 a year.

The trusts of Franklin and Holden didn't turn out exactly as they had planned—Franklin's trust didn't grow adequately, and Holden's trust was converted into annual cash flows. This goes to show that you can't always trust a trust!

Sources: Jake Palmateer, "On the Bright Side: Hartwick College Receives \$9 million Trust," *The Daily Star, Oneonta, NY*, January 22, 2008, <http://thedailystar.com/local/x112892349/On-The-Bright-Side-Hartwick-College-receives-9M-trust/print>; Lewis H. Lapham, "Trust Issues," *Lapham's Quarterly*, Friday December 2, 2011, www.laphamsquarterly.org/essays/trust-issues.php?page51.

28-3 Present Values

Suppose you have some extra money and want to make an investment. A broker offers to sell you a bond that will pay a guaranteed \$115.76 in 3 years. Banks are currently offering a guaranteed 5% interest on 3-year certificates of deposit (CDs), and if you don't buy the bond you will buy a CD. The 5% rate paid on the CD is defined as your **opportunity cost**, or the rate of return you would earn on an alternative investment of similar risk if you don't invest in the security under consideration. Given these conditions, what's the most you should pay for the bond?

28-3a Discounting a Future Value to Find the Present Value

First, recall from the future value example in the last section that if you invested \$100 at 5% in a CD, it would grow to \$115.76 in 3 years. You would also have \$115.76 after 3 years if you bought the bond. Therefore, the most you should pay for the bond is \$100—this is its "fair price," which is also its intrinsic, or fundamental, value. If you could buy the bond for *less than* \$100, then you should buy it rather than invest in the CD. Conversely, if its price were *more than* \$100, you

should buy the CD. If the bond's price were exactly \$100, you should be indifferent between the bond and the CD.

The \$100 is defined as the present value, or PV, of \$115.76 due in 3 years when the appropriate interest rate is 5%. In general, *the present value of a cash flow due N years in the future is the amount that, if it were on hand today, would grow to equal the given future amount.* Because \$100 would grow to \$115.76 in 3 years at a 5% interest rate, \$100 is the present value of \$115.76 due in 3 years at a 5% rate.

Finding present values is called **discounting**, and as previously noted, it is the reverse of compounding: If you know the PV, you can compound it to find the FV; or if you know the FV, you can discount it to find the PV. Indeed, we simply solve Equation 28-1, the formula for the future value, for the PV to produce the present value equation as follows:

$$\text{Compounding to find future values: Future value} = FV_N = PV(1 + I)^N$$

(28-1)

$$\text{Discounting to find present values: Present value} = PV = \frac{FV_N}{(1 + I)^N}$$

(28-3)

The top section of Figure 28-3 shows inputs and a time line for finding the present value of \$115.76 discounted back for 3 years. We first calculate the PV using the step-by-step approach. When we found the FV in the previous section, we worked from left to right, *multiplying* the initial amount and each subsequent amount by $(1 + I)$. To find present values, we work backwards, or from right to left, *dividing* the future value and each subsequent amount by $(1 + I)$, with the present value of \$100 shown in Cell D105. The step-by-step procedure shows exactly what's happening, and that can be quite useful when you are working complex problems or trying to explain a model to others. However, it's inefficient, especially if you are dealing with more than a year or two.

A more efficient procedure is to use the formula approach in Equation 28-3, simply dividing the future value by $(1 + I)^N$. This gives the same result, as we see in Figure 28-3, Cell G107.

Equation 28-2 is actually programmed into financial calculators. As shown in Figure 28-3, Rows 109 to 111, we can find the PV by entering values for $N = 3$, $I/YR = 5$, $PMT = 0$, and $FV = 115.76$, and then pressing the PV key to get -100 .

Excel also has a function that solves Equation 28-3—this is the **PV** function, and it is written as **=PV(I,N,0,FV)**.⁴ Cell E126 shows the inputs to this function. Next, Cell E114 shows the *Excel* function with fixed numbers as inputs, with the actual function and the resulting $-\$100$ in Cell G114. Cell E115 shows the *Excel* function using cell references, with the actual function and the resulting $-\$100$ in Cell G115.

As with the future value calculation, students often wonder why the result of the present value calculation is sometimes positive and sometimes negative. In the algebraic calculations in Rows 105 and 107 the result is $+\$100$, while the result of the calculation using a financial calculator or *Excel's* function in Rows 111 and

WEB

See **Ch28 Tool Kit.xls**
for all calculations.

4. The third entry in the PV function is zero to indicate that there are no intermediate payments in this example.

FIGURE 28-3 Alternative Procedures for Calculating Present Values

	A	B	C	D	E	F	G
97	INPUTS:						
98	Future payment = $CF_N = FV =$		\$115.76				
99	Interest rate = $I =$		5.00%				
100	No. of periods = $N =$		3				
101							
102	Time Line	Periods:	0	1	2	3	
103		Cash flow:	PV = ?	0	0	\$115.76	
104							
105	1. Step-by-Step:		\$100.00	← \$105.00	← \$110.25	← \$115.76	
106							
107	2. Formula: $PV_N = FV/(1+I)^N$			$PV = \$115.76/(1.05)^3$	=		\$100.00
108							
109	Inputs:	3	5		0		115.76
110	3. Financial Calculator:	N	I/YR	PV	PMT	FV	
111	Output:			-\$100.00			
112							
113	4. Excel Spreadsheet:	PV function:	PV =	=PV(I,N,0,FV)			
114		Fixed inputs:	PV =	=PV(0.05,3,0,115.76) =			-\$100.00
115		Cell references:	PV =	=PV(C99,C100,0,C98) =			-\$100.00
116	In the Excel formula, the terms are entered in this sequence: interest, periods, 0 to indicate no periodic						
117	cash flows, and then the FV. The data can be entered as fixed numbers or, better yet, as cell references.						

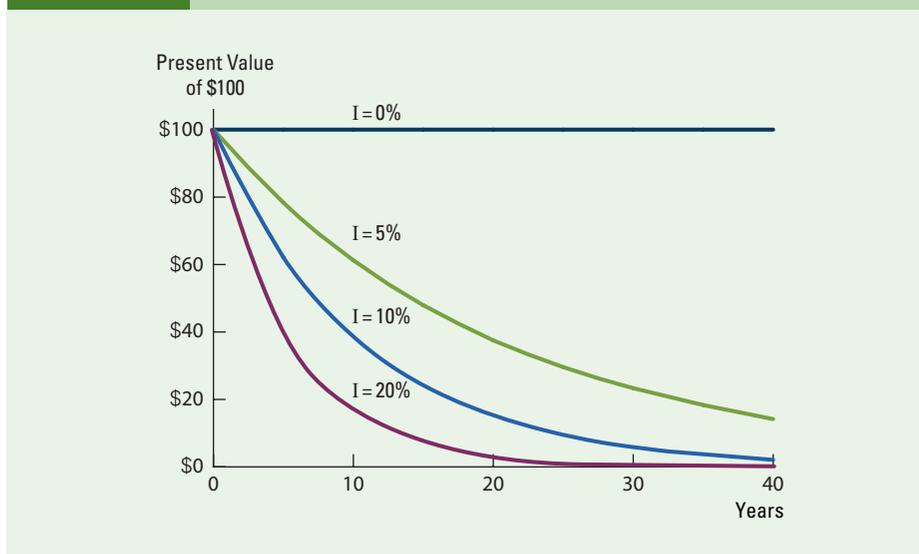
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114 is $-\$100$. Again, the answer is in the signs of a correctly constructed time line. Outflows are negative and inflows are positive. The PV function for Excel and a financial calculator answer the question, “How much must I invest today in order to have available to me a certain amount of money in the future?” If you want to have \$115.76 available in 3 years (an inflow to you, and therefore positive), then you must invest \$100 today (an outflow, and therefore negative). If you use the algebraic functions as in Rows 105 and 107, you must keep track of whether the results of your calculations are inflows or outflows.

The fundamental goal of financial management is to maximize the firm’s intrinsic value, and the intrinsic value of a business (or any asset, including stocks and bonds) is the *present value* of its expected future cash flows. Because present value lies at the heart of the valuation process, we will have much more to say about it in the remainder of this chapter and throughout the book.

28-3b Graphic View of the Discounting Process

Figure 28-4 shows that the present value of a sum to be received in the future decreases and approaches zero as the payment date is extended further and further into the future; it also shows that, the higher the interest rate, the faster the present value falls. At relatively high rates, funds due in the future are worth very little today, and even at relatively low rates present values of sums due in the very distant

FIGURE 28-4 Present Value of \$100 at Various Interest Rates and Time Periods

future are quite small. For example, at a 20% discount rate, \$100 due in 40 years would be worth less than 7 cents today. (However, 1 cent would grow to almost \$1 million in 100 years at 20%.)

WEB

See *Ch28 Tool Kit.xls*
for all calculations.

What is “discounting,” and how is it related to compounding? How is the future value equation (28-1) related to the present value equation (28-3)?

How does the present value of a future payment change as the time to receipt is lengthened? As the interest rate increases?

Suppose a risk-free bond promises to pay \$2,249.73 in 3 years. If the going risk-free interest rate is 4%, how much is the bond worth today? **(\$2,000)**

How much is the bond worth if it matures in 5 rather than 3 years?

(\$1,849.11) If the risk-free interest rate is 6% rather than 4%, how much is the 5-year bond worth today? **(\$1,681.13)**

How much would \$1 million due in 100 years be worth today if the discount rate were 5%? **(\$7,604.49)** What if the discount rate were 20%? **(\$0.0121)**

SELF TEST

28-4 Finding the Interest Rate, *I*

We have used Equations 28-1, 28-2, and 28-3 to find future and present values. Those equations have four variables, and if we know three of them, then we (or our calculator or *Excel*) can solve for the fourth. Thus, if we know PV, *I*, and *N*, we can solve Equation 28-1 for FV, or if we know FV, *I*, and *N*, we can solve Equation 28-3 to find PV. That’s what we did in the preceding two sections.

Now suppose we know PV, FV, and N, and we want to find I. For example, suppose we know that a given security has a cost of \$100 and that it will return \$150 after 10 years. Thus, we know PV, FV, and N, and we want to find the rate of return we will earn if we buy the security. Here's the solution using Equation 28-1 (with FV on the right side of the formula):

$$\begin{aligned} PV(1 + I)^N &= FV \\ \$100(1 + I)^{10} &= \$150 \\ (1 + I)^{10} &= \$150/\$100 \\ (1 + I)^{10} &= 1.5 \\ (1 + I) &= 1.5^{(1/10)} \\ 1 + I &= 1.0414 \\ I &= 0.0414 = 4.14\% \end{aligned}$$

Finding the interest rate by solving the formula takes a little time and thought, but financial calculators and spreadsheets find the answer almost instantly. Here's the calculator setup:

Inputs	10	-100	0	150
	N	I/YR	PV	PMT
Output	4.14			

Enter N = 10, PV = -100, PMT = 0 (because there are no payments until the security matures), and FV = 150. Then, when you press the I/YR key, the calculator gives the answer, 4.14%. Notice that the PV is a negative value because it is a cash outflow (an investment) and the FV is positive because it is a cash inflow (a return of the investment). If you enter both PV and FV as positive numbers (or both as negative numbers), you will get an error message rather than the answer.

In *Excel*, the RATE function can be used to find the interest rate: =RATE(N,PMT,PV,FV). For this example, the interest rate is found as =RATE(10,0,-100,150) = 0.0414 = 4.14%. See the file *Ch28 Tool Kit.xls* on the textbook's Web site for an example.

WEB

See *Ch28 Tool Kit.xls* for all calculations.

SELF TEST

Suppose you can buy a U.S. Treasury bond that makes no payments until the bond matures 10 years from now, at which time it will pay you \$1,000.⁵ What interest rate would you earn if you bought this bond for \$585.43? **(5.5%)** What rate would you earn if you could buy the bond for \$550? **(6.16%)** For \$600? **(5.24%)**

Microsoft earned \$0.33 per share in 1997. Fourteen years later, in 2011, it earned \$2.75. What was the growth rate in Microsoft's earnings per share (EPS) over the 14-year period? **(16.35%)** If EPS in 2011 had been \$2.00 rather than \$2.75, what would the growth rate have been? **(13.73%)**

5. This is a STRIP bond, which we explain in Chapter 4.

28-5 Finding the Number of Years, N

We sometimes need to know how long it will take to accumulate a specific sum of money, given our beginning funds and the rate we will earn. For example, suppose we now have \$500,000 and the interest rate is 4.5%. How long will it be before we have \$1 million?

Here's Equation 28-1, showing all the known variables:

$$\$1,000,000 = \$500,000(1 + 0.045)^N$$

(28-1)

We need to solve for N, and we can use three procedures: a financial calculator, *Excel* (or some other spreadsheet), or by working with natural logs. As you might expect, the calculator and spreadsheet approaches are easier.⁶ Here's the calculator setup:

Inputs		4.5	-500000	0	1000000
	N	I/YR	PV	PMT	FV
Output	15.7473				

Enter I/YR = 4.5, PV = -500000, PMT = 0, and FV = 1000000. We press the N key to get the answer, 15.7473 years. In *Excel*, we would use the NPER function: =NPER(I,PMT,PV,FV). Inserting data, we have =NPER(0.045,0,-500000,1000000) = 15.7473. The chapter's tool kit, *Ch28 Tool Kit.xls*, shows this example.



See *Ch28 Tool Kit.xls*
for all calculations.

How long would it take \$1,000 to double if it were invested in a bank that pays 6% per year? **(11.9 years)** How long would it take if the rate were 10%? **(7.27 years)**

A company's 2015 earnings per share were \$2.75, and its growth rate during the prior 14 years was 16.35% per year. If that growth rate were maintained, how long would it take for EPS to double? **(4.58 years)**



28-6 Perpetuities

The previous sections examined the relationship between the present value and the future value of a single payment at a fixed point in time. However, some securities promise to make payments forever. For example, preferred stock, which we discuss in Chapter 8, promises to pay a dividend forever. Another "forever" security originated in the mid-1700s when the British government issued some bonds

6. Here's the setup for the log solution. First, transform Equation 28-1 as indicated, then find the natural logs using a financial calculator, and then solve for N:

$$\$1,000,000 = \$500,000(1 + 0.045)^N$$

$$2 = (1 + 0.045)^N$$

$$\ln(2) = N[\ln(1.045)]$$

$$N = 0.6931/0.0440 = 15.7473 \text{ years}$$

that never matured and whose proceeds were used to pay off other British bonds. Because this action consolidated the government's debt, the new bonds were called "consols." The term stuck, and now any bond that promises to pay interest perpetually is called a **consol**, or a perpetuity. The interest rate on the consols was 2.5%, so a consol with a face value of £1,000 would pay £25 per year in perpetuity (£ is the currency symbol for a British pound).

A consol, or **perpetuity**, is simply an annuity whose promised payments extend out forever. Since the payments go on forever, you can't apply the step-by-step approach. However, it's easy to find the PV of a perpetuity with the following formula:⁷

(28-4)

$$\text{PV of a perpetuity} = \frac{\text{PMT}}{I}$$

We can use Equation 28-4 to find the value of a British consol with a face value of £1,000 that pays £25 per year in perpetuity. The answer depends on the interest rate being earned on investments of comparable risk at the time the consol is being valued. Originally, the "going rate" as established in the financial marketplace was 2.5%, so originally the consol's value was £1,000:

$$\text{Consol's value}_{\text{Originally}} = £25/0.025 = £1,000$$

The annual payment is still £25 today, but the going interest rate has risen to about 5.2%, causing the consol's value to fall to £480.77:

$$\text{Consol's value}_{\text{Today}} = £25/0.052 = £480.77$$

Note, though, that if interest rates decline in the future, say to 2%, then the value of the consol will rise to £1,250.00:

$$\text{Consol's value if rates decline to } 2\% = £25/0.02 = £1,250.00$$

These examples demonstrate an important point: *When interest rates change, the prices of outstanding bonds also change, but inversely to the change in rates. Thus, bond prices decline if rates rise, and prices increase if rates fall. This holds for all bonds, both consols and those with finite maturities.* We discuss this point in more detail in Chapter 4, where we cover bonds in depth.

SELF TEST

What is the present value of a perpetuity that pays £1,000 per year, beginning 1 year from now, if the appropriate interest rate is 5%? **(£20,000)** What would the value be if the perpetuity began its payments immediately? **(£21,000)** (*Hint: Just add the £1,000 to be received immediately to the formula value of the annuity.*)

Do bond prices move directly or inversely with interest rates—that is, what happens to the value of a bond if interest rates increase or decrease?

7. Here is an intuitive explanation for Equation 28-4. Suppose you deposit an amount equal to PV in a bank that pays an interest rate of I. Each year you would be able to withdraw an amount equal to $I \times \text{PV}$. If you left your deposit in forever, you could withdraw a payment of $I \times \text{PV}$ forever: $\text{PMT} = I \times \text{PV}$. Rearranging, we get Equation 28-4. This is only an intuitive explanation, so see *Web Extension 28B* on the textbook's Web site for a mathematical derivation of the perpetuity formula.

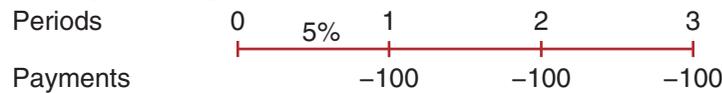
28-7 Annuities

Thus far, we have dealt with single payments, or “lump sums.” However, assets such as bonds provide a series of cash inflows over time, and obligations such as auto loans, student loans, and mortgages call for a series of payments. If the payments are equal and are made at fixed intervals, then we have an **annuity**. For example, \$100 paid at the end of each of the next 3 years is a 3-year annuity.

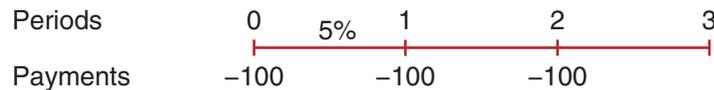
If payments occur at the *end* of each period, then we have an **ordinary annuity**, which is also called a **deferred annuity**. Payments on mortgages, car loans, and student loans are generally made at the ends of the periods and thus are ordinary annuities. If the payments are made at the *beginning* of each period, then we have an **annuity due**. Rental lease payments, life insurance premiums, and lottery payoffs (if you are lucky enough to win one!) are examples of annuities due. Ordinary annuities are more common in finance, so when we use the term “annuity” in this book, you may assume that the payments occur at the ends of the periods unless we state otherwise.

Next we show the time lines for a \$100, 3-year, 5%, ordinary annuity and for the same annuity on an annuity due basis. With the annuity due, each payment is shifted back (to the left) by 1 year. In our example, we assume that a \$100 payment will be made each year, so we show the payments with minus signs.

Ordinary Annuity:



Annuity Due:



As we demonstrate in the following sections, we can find an annuity’s future value, present value, the interest rate built into the contracts, how long it takes to reach a financial goal using the annuity, and, if we know all of those values, the size of the annuity payment. Keep in mind that annuities must have *constant payments* and a *fixed number of periods*. If these conditions don’t hold, then the series is not an annuity.

What’s the difference between an ordinary annuity and an annuity due?
 Why should you prefer to receive an *annuity due* with payments of \$10,000 per year for 10 years than an otherwise similar *ordinary annuity*?

SELF TEST

28-8 Future Value of an Ordinary Annuity

Consider the ordinary annuity whose time line was shown previously, where you deposit \$100 at the *end* of each year for 3 years and earn 5% per year. Figure 28-5 shows how to calculate the **future value of the annuity**, FVA_N , using the same approaches we used for single cash flows.

FIGURE 28-5 Summary: Future Value of an Ordinary Annuity

	A	B	C	D	E	F	G
243	INPUTS:						
244	Payment amount = PMT =		-\$100				
245	Interest rate = I =		5.00%				
246	No. of periods = N =		3				
247							
248	1. Step-by-Step:	Periods:	0	1	2	3	
249		Cash flow:		-\$100	-\$100	-\$100	
250				↓	↓	↓	
251				↓	↓	\$100.00	
252	Multiply each payment by			↓	↳ →	\$105.00	
253	(1+I) ^{N-t} and sum these FVs to			↳ → → → → →	↳ → → → → →	\$110.25	
254	find FVA _N :					\$315.25	
255							
256	2. Formula:						
257							
258							
259							
260							
261		Inputs:	3	5	0	-100	
262	3. Financial Calculator:		N	I/YR	PV	PMT	FV
263		Output:					\$315.25
264							
265	4. Excel Spreadsheet:	FV function:	FVA _N =	=FV(I,N,PMT,PV)			
266		Fixed inputs:	FVA _N =	=FV(0.05,3,-100,0)			\$315.25
267		Cell references:	FVA _N =	=FV(C245,C246,C244,0)			\$315.25

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As shown in the step-by-step section of Figure 28-5, we compound each payment out to Time 3, and then sum those compounded values in Cell F254 to find the annuity's FV, $FVA_3 = \$315.25$ (we ignore the negative signs in the time line to be consistent with the results from the other methods). The first payment earns interest for two periods, the second for one period, and the third earns no interest because it is made at the end of the annuity's life. This approach is straightforward, but if the annuity extends out for many years, it is cumbersome and time-consuming.

As you can see from the time line diagram, with the step-by-step approach we apply the following equation with $N = 3$ and $I = 5\%$:

$$\begin{aligned} FVA_N &= PMT(1 + I)^{N-1} + PMT(1 + I)^{N-2} + PMT(1 + I)^{N-3} \\ &= \$100(1.05)^2 + \$100(1.05)^1 + \$100(1.05)^0 \\ &= \$315.25 \end{aligned}$$

For the general case, the future value of an annuity is:

$$FVA_N = PMT(1 + I)^{N-1} + PMT(1 + I)^{N-2} + PMT(1 + I)^{N-3} + \dots + PMT(1 + I)^0$$

As shown in *Web Extension 28B* on the textbook's Web site, the future value of an annuity can be written as follows:⁸

$$FVA_N = PMT \left[\frac{(1 + I)^N}{I} - \frac{1}{I} \right]$$

(28-5)

Using Equation 28-5, the future value of the annuity is found to be \$315.25:

$$FVA_3 = \$100 \left[\frac{(1 + 0.05)^3}{0.05} - \frac{1}{0.05} \right] = \$315.25$$

As you might expect, annuity problems can be solved easily using a financial calculator or a spreadsheet, most of which have the following formula built into them:

$$PV(1 + I)^N + PMT \left[\frac{(1 + I)^N}{I} - \frac{1}{I} \right] + FV = 0$$

(28-6)

The procedure when dealing with annuities is similar to what we have done thus far for single payments, but the presence of recurring payments means that we must use the PMT key. Here's the calculator setup for our illustrative annuity:

Inputs	3	5	0	-100	End Mode
	N	I/YR	PV	PMT	FV
Output					315.25

We enter $PV = 0$ because we start off with nothing, and we enter $PMT = -100$ because we will deposit this amount in the account at the end of each of the 3 years. The interest rate is 5%, and when we press the FV key we get the answer, $FVA_3 = 315.25$.

Because this is an ordinary annuity, with payments coming at the *end* of each year, we must set the calculator appropriately. As noted earlier, most calculators “come out of the box” set to assume that payments occur at the end of each period—that is, to deal with ordinary annuities. However, there is a key that enables us to switch between ordinary annuities and annuities due. For ordinary annuities, the designation “End Mode” or something similar is used, while for annuities due the designator is “Begin,” “Begin Mode,” “Due,” or something similar. If you make

8. Section 28-6 shows that the present value of an infinitely long annuity, called a perpetuity, is equal to PMT/I . The cash flows of an ordinary annuity of N periods are equal to the cash flows of a perpetuity minus the cash flows of a perpetuity that begins at year $N + 1$. Therefore, the future value of an N -period annuity is equal to the future value (as of year N) of a perpetuity minus the value (as of year N) of a perpetuity that begins at year $N + 1$. See *Web Extension 28B* on the textbook's Web site for details regarding derivations of Equation 28-5.

THE POWER OF COMPOUND INTEREST

Assume that you are 26 and just received your MBA. After reading the introduction to this chapter, you decide to start investing in the stock market for your retirement. Your goal is to have \$1 million when you retire at age 65. Assuming you earn 10% annually on your stock investments, how much must you invest at the end of each year in order to reach your goal?

The answer is \$2,491, but this amount depends critically on the return your investments earn. If your return drops to 8%, the required annual contribution would rise to \$4,185. On the other hand, if the return rises to 12%, you would need to put away only \$1,462 per year.

What if you are like most 26-year-olds and wait until later to worry about retirement? If you wait until age 40,

you will need to save \$10,168 per year to reach your \$1 million goal, assuming you can earn 10%, but \$13,679 per year if you earn only 8%. If you wait until age 50 and then earn 8%, the required amount will be \$36,830 per year!

Although \$1 million may seem like a lot of money, it won't be when you get ready to retire. If inflation averages 5% a year over the next 39 years, then your \$1 million nest egg would be worth only \$149,148 in today's dollars. If you live for 20 years after retirement and earn a real 3% rate of return, your annual retirement income in today's dollars would be only \$9,733 before taxes. So, after celebrating your graduation and new job, start saving!

WEB

See *Ch28 Tool Kit.xls* for all calculations.

a mistake and set your calculator on Begin Mode when working with an ordinary annuity, then each payment will earn interest for 1 extra year, which will cause the compounded amounts, and thus the FVA, to be too large.

The spreadsheet approach uses *Excel's* FV function, `=FV(I,N,PMT,PV)`. In our example, we have `=FV(0.05,3,-100,0)`, and the result is again \$315.25.

SELF TEST

For an ordinary annuity with 5 annual payments of \$100 and a 10% interest rate, for how many years will the *first payment* earn interest, and what is the compounded value of this payment at the end? **(4 years, \$146.41)**
Answer these questions for the fifth payment. **(0 years, \$100)**

Assume that you plan to buy a condo 5 years from now, and you estimate that you can save \$2,500 per year toward a down payment. You plan to deposit the money in a bank that pays 4% interest, and you will make the first deposit at the end of this year. How much will you have after 5 years? **(\$13,540.81)** How would your answer change if the bank's interest rate were increased to 6%, or decreased to 3%? **(\$14,092.73; \$13,272.84)**

28-9 Future Value of an Annuity Due

Because each payment occurs one period earlier with an annuity due, the payments will all earn interest for one additional period. Therefore, the FV of an annuity due will be greater than that of a similar ordinary annuity.

If you followed the step-by-step procedure, you would see that our illustrative annuity due has a FV of \$331.01 versus \$315.25 for the ordinary annuity.

See *Ch28 Tool Kit.xls* on the textbook's Web site for a summary of future value calculations.

With the formula approach, we first use Equation 28-5, but because each payment occurs one period earlier, we multiply the Equation 28-5 result by $(1 + I)$:

$$FVA_{\text{due}} = FVA_{\text{ordinary}}(1 + I)$$

(28-7)

Thus, for the annuity due, $FVA_{\text{due}} = \$315.25(1.05) = \331.01 , which is the same result as found with the step-by-step approach.

With a calculator, we input the variables just as we did with the ordinary annuity, but we now set the calculator to Begin Mode to get the answer, \$331.01:

Inputs	3	5	0	-100	Begin Mode
	N	I/YR	PV	PMT	
Output					FV 331.01

In *Excel*, we still use the FV function, but we must indicate that we have an annuity due. The function is `=FV(I,N,PMT,PV,Type)`, where "Type" indicates the type of annuity. If Type is omitted, then *Excel* assumes that it is 0, which indicates an ordinary annuity. For an annuity due, Type = 1. As shown in *Ch28 Tool Kit.xls*, the function is `=FV(0.05,3,-100,0,1) = $331.01`.

WEB

See *Ch28 Tool Kit.xls* for all calculations.

Why does an annuity due always have a higher future value than an ordinary annuity?

If you know the value of an ordinary annuity, explain why you could find the value of the corresponding annuity due by multiplying by $(1 + I)$.

Assume that you plan to buy a condo 5 years from now and that you need to save for a down payment. You plan to save \$2,500 per year, with the first payment being made *immediately* and deposited in a bank that pays 4%. How much will you have after 5 years? **(\$14,082.44)** How much would you have if you made the deposits at the *end* of each year? **(\$13,540.81)**

SELF TEST

28-10 Present Value of Ordinary Annuities and Annuities Due

The present value of any annuity, PVA_N , can be found using the step-by-step, formula, calculator, or spreadsheet methods. We begin with ordinary annuities.

28-10a Present Value of an Ordinary Annuity

See Figure 28-6 for a summary of the different approaches for calculating the present value of an ordinary annuity.

FIGURE 28-6 Summary: Present Value of an Ordinary Annuity

	A	B	C	D	E	F	G
313	INPUTS:						
314	Payment amount = PMT =		-\$100				
315	Interest rate = I =		5.00%				
316	No. of periods = N =		3				
317							
318	1. Step-By-Step:	Periods:	0	1	2	3	
319		Cash flow:		-\$100	-\$100	-\$100	
320				↓	↓	↓	
321			\$95.24	← ← ↓	↓	↓	
322	Divide each payment by		\$90.70	← ← ← ← ←	← ← ↓	↓	
323	(1+I) ^t and sum these PVs to		\$86.38	← ← ← ← ←	← ← ← ← ←	← ← ↓	
324	find PVA _N :		\$272.32				
325							
326	2. Formula:						
327							
328							
329							
330							
331							
332		Inputs:	3	5		-100	0
333	3. Financial Calculator:		N	I	PV	PMT	FV
334		Output:			272.32		
335							
336	4. Excel Spreadsheet:	PV function:	PVA _N =	=PV(I,N,PMT,FV)			
337		Fixed inputs:	PVA _N =	=PV(0.05,3,-100,0) = \$272.32			
338		Cell references:	PVA _N =	=PV(C315,C316,C314,0) = \$272.32			

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See *Ch28 Tool Kit.xls* for all calculations.

As shown in the step-by-step section of Figure 28-6, we discount each payment back to Time 0, and then sum those discounted values to find the annuity's PV, $PVA_3 = \$272.32$ (we ignore the negative signs in the time line to be consistent with the results from the other methods). This approach is straightforward, but if the annuity extends out for many years, it is cumbersome and time-consuming.

The time line diagram shows that with the step-by-step approach we apply the following equation with $N = 3$ and $I = 5\%$:

$$PVA_N = PMT/(1 + I)^1 + PMT/(1 + I)^2 + \dots + PMT/(1 + I)^N$$

The present value of an annuity can be written as:⁹

(28-8)

$$PVA_N = PMT \left[\frac{1}{I} - \frac{1}{I(1 + I)^N} \right]$$

9. See *Web Extension 28B* on the textbook's Web site for details of this derivation.

For our example annuity, the present value is

$$PVA_3 = \$100 \left[\frac{1}{0.05} - \frac{1}{0.05(1 + 0.05)^3} \right] = \$272.32$$

Financial calculators are programmed to solve Equation 28-7, so we input the variables and press the PV key, *first making sure the calculator is set to End Mode*. The calculator setup is shown below:

Inputs	3	5	-100	0	End Mode (Ordinary Annuity)
	N	I/YR	PV	PMT	FV
Output			272.32		

The last section of Figure 28-6 shows the spreadsheet solution using *Excel's* built-in PV function: =PV(I,N,PMT,FV). In our example, we have =PV(0.05,3,-100,0) with a resulting value of \$272.32.

WEB

See *Ch28 Tool Kit.xls* for all calculations.

28-10b Present Value of Annuities Due

Because each payment for an annuity due occurs one period earlier, the payments will all be discounted for one less period. Therefore, the PV of an annuity due must be greater than that of a similar ordinary annuity.

If you went through the step-by-step procedure, you would see that our example annuity due has a PV of \$285.94 versus \$272.32 for the ordinary annuity. See *Ch28 Tool Kit.xls* for this and the other calculations.

With the formula approach, we first use Equation 28-8 to find the value of the ordinary annuity and then, because each payment now occurs one period earlier, we multiply the Equation 28-8 result by (1 + I):

$$PVA_{\text{due}} = PVA_{\text{ordinary}}(1 + I) \tag{28-9}$$

$$PVA_{\text{due}} = \$272.32(1.05) = \$285.94$$

With a financial calculator, the inputs are the same as for an ordinary annuity, except you must set the calculator to Begin Mode:

Inputs	3	5	-100	0	Begin Mode (Annuity Due)
	N	I/YR	PV	PMT	FV
Output			285.94		

WEB

See *Ch28 Tool Kit.xls* for all calculations.

In *Excel*, we again use the PV function, but now we must indicate that we have an annuity due. The function is now =PV(I,N,PMT,FV,Type), where "Type" is the type of annuity. If Type is omitted then *Excel* assumes that it is 0, which indicates

VARIABLE ANNUITIES: GOOD OR BAD?

Retirees appreciate stable, predictable income, so they often buy annuities. Insurance companies have been the traditional suppliers, using the payments they receive to buy high-grade bonds, whose interest is then used to make the promised payments. Such annuities were quite safe and stable and provided returns of around 7.5%. However, returns on stocks (dividends plus capital gains) have historically exceeded bonds' returns (interest). Therefore, some insurance companies in the 1990s began to offer *variable annuities*, which were backed by stocks instead of bonds. If stocks earned in the future as much as they had in the past, then variable annuities could offer returns of about 9%; this is better than the return on fixed-rate annuities. If stock returns turned out to be lower in the future than they had been in the past (or even had negative returns), then the variable annuities promised a *guaranteed minimum payment* of about 6.5%. Variable annuities appealed to many retirees, so companies that offered them had a significant competitive advantage.

The insurance company that pioneered variable annuities, The Hartford Financial Services Group, tried to hedge its position with derivatives that paid off if stocks went down. But like so many other derivatives-based risk management programs, this one went awry in 2008 because stock losses exceeded the assumed worst-case scenario. The Hartford, which was founded in 1810 and was one of the oldest and largest U.S. insurance companies at the beginning of 2008, saw its stock price fall from \$85.54 per share to \$4.16. Because of the general stock market crash, investors feared that The Hartford would be unable to make good on its variable annuity promises, and this would lead to bankruptcy. The company was bailed out by the economic stimulus package, but this venerable old firm will never be the same again.

Source: Leslie Scism and Liam Plevin, "Hartford Aims to Take Risk Out of Annuities," *Online Wall Street Journal*, January 13, 2009.

an ordinary annuity; for an annuity due, Type = 1. As shown in *Ch28 Tool Kit.xls*, the function for this example is $=PV(0.05,3,-100,0,1) = \$285.94$.

SELF TEST

Why does an annuity due have a higher present value than an ordinary annuity?

If you know the present value of an ordinary annuity, what's an easy way to find the PV of the corresponding annuity due?

What is the PVA of an ordinary annuity with 10 payments of \$100 if the appropriate interest rate is 10%? **(\$614.46)** What would the PVA be if the interest rate were 4%? **(\$811.09)** What if the interest rate were 0%? **(\$1,000.00)** What would the PVAs be if we were dealing with annuities due? **(\$675.90, \$843.53, and \$1,000.00)**

Assume that you are offered an annuity that pays \$100 at the end of each year for 10 years. You could earn 8% on your money in other equally risky investments. What is the most you should pay for the annuity? **(\$671.01)** If the payments began immediately, then how much would the annuity be worth? **(\$724.69)**

28-11 Finding Annuity Payments, Periods, and Interest Rates

In the three preceding sections, we discussed how to find the FV and PV of ordinary annuities and annuities due, using these four methods: step-by-step, formula, financial calculator, and *Excel*. Five variables are involved—N, I, PMT, FV, and PV—and if you know any four, you can find the fifth by solving either Equation 28-4 (28-6 for annuities due) or 28-7 (28-8 for annuities due). However, a trial-and-error procedure is generally required to find N or I, and that can be quite tedious if performed manually. Therefore, we discuss only the financial calculator and spreadsheet approaches for finding N and I.

28-11a Finding Annuity Payments, PMT

We need to accumulate \$10,000 and have it available 5 years from now. We can earn 6% on our money. Thus, we know that FV = 10,000, PV = 0, N = 5, and I/YR = 6. We can enter these values in a financial calculator and then press the PMT key to find our required deposits. However, the answer depends on whether we make deposits at the end of each year (ordinary annuity) or at the beginning (annuity due), so the mode must be set properly. Here are the results for each type of annuity:

Inputs	5	6	0		10000	End Mode (Ordinary Annuity)
	N	I/YR	PV	PMT	FV	
Output				-1773.96		

Inputs	5	6	0		10000	Begin Mode (Annuity Due)
	N	I/YR	PV	PMT	FV	
Output				-1673.55		

Thus, you must put away \$1,773.96 per year if you make payments at the *end* of each year, but only \$1,673.55 if the payments begin *immediately*. Finally, note that the required payment for the annuity due is the ordinary annuity payment divided by $(1 + I)$: $\$1,773.96/1.06 = \$1,673.55$.

Excel can also be used to find annuity payments, as shown below for the two types of annuities. For end-of-year (ordinary) annuities, “Type” can be left blank or a 0 can be inserted. For beginning-of-year annuities (annuities due), the same function is used but now Type is designated as 1. Here is the setup for the two types of annuities.

Function: = PMT(I,N,PV,FV,Type)

Ordinary annuity: = PMT(0.06,5,0,10000) = - \$1,773.96

Annuity due: = PMT(0.06,5,0,10000,1) = - \$1,673.55

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See **Ch28 Tool Kit.xls**
for all calculations.

28-11b Finding the Number of Periods, N

Suppose you decide to make end-of-year deposits, but you can save only \$1,200 per year. Again assuming that you would earn 6%, how long would it take you to reach your \$10,000 goal? Here is the calculator setup:

Inputs		6	0	-1200	10000	End Mode
	N	I/YR	PV	PMT	FV	
Output	6.96					

With these smaller deposits, it would take 6.96 years, not 5 years, to reach the \$10,000 target. If you began the deposits immediately, then you would have an annuity due and N would be slightly less, 6.63 years.

With *Excel*, you can use the NPER function: `=NPER(I,PMT,PV,FV,Type)`. For our ordinary annuity example, Type is left blank (or 0 is inserted) and the function is `=NPER(0.06,-1200,0,10000) = 6.96`. If we put in 1 for Type, we would find $N = 6.63$.

WEB

See *Ch28 Tool Kit.xls* for all calculations.

28-11c Finding the Interest Rate, I

Now suppose you can save only \$1,200 annually, but you still need to have the \$10,000 in 5 years. What rate of return would you have to earn to reach your goal? Here is the calculator setup:

Inputs	5	0	-1200	10000	End Mode
	N	I/YR	PV	PMT	FV
Output		25.78			

Thus, you would need to earn a whopping 25.78%! About the only way to earn such a high return would be either to invest in speculative stocks or head to a Las Vegas casino. Of course, speculative stocks and gambling aren't like making deposits in a bank with a guaranteed rate of return, so there would be a high probability that you'd end up with nothing. So, you should probably save more, lower your \$10,000 target, or extend your time horizon. It might be appropriate to seek a somewhat higher return, but trying to earn 25.78% in a 6% market would involve speculation, not investing.

In *Excel*, you can use the RATE function: `=RATE(N,PMT,PV,FV,Type)`. For our example, the function is `=RATE(5,-1200,0,10000) = 0.2578 = 25.78%`. If you decide to make the payments beginning immediately then the required rate of return would decline sharply, to 17.54%.

SELF TEST

You just inherited \$100,000 and invested it at 7% per year. How large a withdrawal could you make at the *end* of each of the next 10 years and end up with zero? (**\$14,237.75**) How would your answer change if you made withdrawals at the *beginning* of each year? (**\$13,306.31**)

(Continued)

USING THE INTERNET FOR PERSONAL FINANCIAL PLANNING

How good are your financial planning skills? For example, should you buy or lease a car? How much and how soon should you begin to save for your children's education? How expensive a house can you afford? Should you refinance your home mortgage? How much must you save each year if you are to retire comfortably? The answers to these questions are often complicated and depend on a number of factors, such as projected housing and education costs, interest rates, inflation, expected family income, and stock market returns.

Fortunately, you should be able to use time value of money concepts and online resources to begin developing your financial plan. In addition to the online data sources described in Chapter 6, an excellent source of information is available at www.smartmoney.com. *Smartmoney* is a personal finance magazine produced by the publishers of *The Wall Street Journal*. If you go to *Smartmoney's* Web site, you will find a section entitled "Tools." This section has a number of financial calculators, spreadsheets, and descriptive materials that cover a wide range of personal finance issues.

If you have \$100,000 that is invested at 7% and you wanted to withdraw \$10,000 at the end of each year, how long will your funds last? **(17.8 years)**
How long would they last if you earned 0%? **(10 years)**
How long would they last if you earned the 7% but limited your withdrawals to \$7,000 per year? **(forever)**

Your uncle named you as the beneficiary of his life insurance policy. The insurance company gives you a choice of \$100,000 today or a 12-year annuity of \$12,000 at the end of each year. What rate of return is the insurance company offering? **(6.11%)**

You just inherited an annuity that will pay you \$10,000 per year for 10 years, and you receive the first payment today. A professional investor offers to give you \$60,000 for the annuity. If you sell it to him, what rate of return will he earn on the investment? **(13.70%)** If you think a "fair" rate of return would be 6%, how much should you ask for the annuity? **(\$78,016.92)**

28-12 Uneven, or Irregular, Cash Flows

The definition of an annuity includes the term *constant payment*, which suggests that annuities involve a set of identical payments over a given number of periods. Although many financial decisions do involve constant payments, many others involve cash flows that are **uneven** or **irregular**. For example, the dividends on common stocks are typically expected to increase over time, and the investments that companies make in new products, expanded production capacity, and replacement machinery almost always generate cash flows that vary from year to year. Throughout the book, we use the term *payment* (PMT) in situations where the cash flows are constant and thus an annuity is involved; if different cash flows occur in different time periods, t , then we use the term CF_t to designate the *cash flow* in period t .

There are two important classes of uneven cash flows: (1) those in which the cash flow stream consists of a series of annuity payments plus an additional final lump sum in Year N, and (2) all other uneven streams. Bonds are an instance of the first type, while stocks and capital investments illustrate the second type. Here's an example of each type.

Stream 1. Annuity plus an additional final payment:

Periods	0	1	2	3	4	5
	----- I = 12% -----					
Cash flows	0	100	100	100	100	100
						<u>1,000</u>
						<u>1,100</u>

Stream 2. Irregular cash flow stream:

Periods	0	1	2	3	4	5
	----- I = 12% -----					
Cash flows	0	100	300	300	300	500

Equation 28-10 can be used, following the step-by-step procedure, to find the PV of either stream. However, as we shall see, the solution process differs significantly for the two types.

(28-10)

$$PV = \frac{CF_1}{(1 + I)^1} + \frac{CF_2}{(1 + I)^2} + \dots + \frac{CF_N}{(1 + I)^N} = \sum_{t=1}^N \frac{CF_t}{(1 + I)^t}$$

28-12a Annuity Plus Additional Final Payment

First, consider Stream 1 and notice that it is a 5-year, 12%, ordinary annuity plus a final payment of \$1,000. We can find the PV of the annuity, find the PV of the final payment, and then sum them to get the PV of the stream. Financial calculators are programmed to do this for us—we use all five time value of money (TVM) keys, entering the data for the four known values as shown below, and then pressing the PV key to get the answer, \$927.90:

Inputs	5	12		100	10000
	N	I/YR	PV	PMT	FV
Output			-927.90		

Similarly, we could use Excel's PV function: =PV(I,N,PMT,FV) = PV(0.12,5,100,1000) = -\$927.90. Note that the process is similar to that for annuities, except we now have a nonzero value for FV.

FIGURE 28-7 Present Value of an Irregular Cash Flow Stream

	A	B	C	D	E	F	G	
466	INPUTS:							
467	Interest rate = I =		12%					
468								
469	1. Step-by-step:							
470	Periods:	0	1	2	3	4	5	
471	Cash Flows:	\$0.00	\$100.00	\$300.00	\$300.00	\$300.00	\$500.00	
472	PVs of the CFs:		\$89.29	\$239.16	\$213.53	\$190.66	\$283.71	
473								
474	PV of the Irregular CF Stream = Sum of the Individual PVs =				\$1,016.35			
475								
476								
477	2. Calculator:	You could enter the cash flows into the cash flow register of a financial calculator, enter I/YR, and then press the NPV key to find the answer.					\$1,016.35	
478								
479	3. Excel Spreadsheet:	NPV Function:	NPV =	= NPV(I,CFs)				
480		Fixed inputs:	NPV =	=NPV(0.12,100,300,300,300,500)			\$1,016.35	
481		Cell references:	NPV =	=NPV(C467,C471:G471)			\$1,016.35	
482	The Excel formula ignores the initial cash flow (in Year 0). When entering a cash flow range, Excel assumes that							
483	the first value occurs at the end of the first year. As we will see later, if there is an initial cash flow, it must be							
484	added separately to complete the NPV formula result. Notice too that you can enter cash flows one-by-one, but if							
485	the cash flows appear in consecutive cells, you can enter the cell range, as we did here.							

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28-12b Irregular Cash Flow Stream

Now consider the irregular stream, which is analyzed in Figure 28-7. The top section shows the basic time line, which contains the inputs, and we first use the step-by-step approach to find $PV = \$1,016.35$. Note that we show the PV of each cash flow directly below the cash flow, and then we sum these PVs to find the PV of the stream. This setup saves space as compared with showing the individual PVs in a column, and it is also transparent and thus easy to understand.

Now consider the financial calculator approach. The cash flows don't form an annuity, so you can't use the annuity feature on the calculator. You could, of course, use the calculator in the step-by-step procedure, but financial calculators have a feature—the cash flow register—that allows you to find the present value more efficiently. First, you input the individual cash flows, in chronological order, into the cash flow register.¹⁰ Cash flows are designated CF_0 , CF_1 , CF_2 , CF_3 , and so on, up to the last cash flow, CF_N . Next, you enter the interest rate, I/YR. At this point, you have substituted in all the known values of Equation 28-10, so when you press the NPV key you get the PV of the stream. The calculator finds the PV of each cash flow and sums them to find the PV of the entire stream. To input the cash flows for this problem, enter 0 (because



See **Ch28 Tool Kit.xls** for all calculations.

10. These instructions are for the HP 10bII+, but most other financial calculators work in a similar manner.

$CF_0 = 0$), 100, 300, 300, 300, and 500 in that order into the cash flow register, enter $I/YR = 12$, and then press NPV to obtain the answer, \$1,016.35.

Two points should be noted. First, when dealing with the cash flow register, the calculator uses the term “NPV” rather than “PV.” The N stands for “net,” so NPV is the abbreviation for “net present value,” which is simply the net present value of a series of positive and negative cash flows, including any cash flow at time zero. The NPV function will be used extensively when we get to capital budgeting, where CF_0 is generally the cost of the project.

The second point to note is that repeated cash flows with identical values can be entered into the cash flow register more efficiently on some calculators by using the N_j key. In this illustration, you would enter $CF_0 = 0$, $CF_1 = 100$, $CF_2 = 300$, $N_j = 3$ (which tells the calculator that the 300 occurs 3 times), and $CF_5 = 500$.¹¹ Then enter $I = 12$, press the NPV key, and 1,016.35 will appear in the display. Also, note that numbers entered into the cash flow register remain in the register until they are cleared. Thus, if you previously worked a problem with eight cash flows, and then moved to one with only four cash flows, the calculator would simply add the cash flows from the second problem to those of the first problem, and you would get an incorrect answer. Therefore, you must be sure to clear the cash flow register before starting a new problem.

Spreadsheets are especially useful for solving problems with uneven cash flows. You enter the cash flows in the spreadsheet as shown in Figure 28-7 on Row 471. To find the PV of these cash flows without going through the step-by-step process, you would use the NPV function. First put the cursor on the cell where you want the answer to appear, Cell G481, click the function wizard, choose Financial, scroll down to NPV, and click OK to get the dialog box. Then enter C467 (or 0.12) for Rate and enter either the individual cash flows or the range of cells containing the cash flows, C471:G471, for Value 1. Be very careful when entering the range of cash flows. With a financial calculator, you begin by entering the Time-0 cash flow. With *Excel*, you do *not* include the Time-0 cash flow; instead, you begin with the Time-1 cash flow. Now, when you click OK, you get the PV of the stream, \$1,016.35. Note that you can use the PV function if the payments are constant, but you must use the NPV function if the cash flows are not constant. Finally, note that *Excel* has a major advantage over financial calculators in that you can see the cash flows, which makes it easy to spot data-entry errors. With a calculator, the numbers are buried in the machine, making it harder to check your work.

SELF TEST

Could you use Equation 28-3, once for each cash flow, to find the PV of an uneven stream of cash flows?

What is the present value of a 5-year ordinary annuity of \$100 plus an additional \$500 at the end of Year 5 if the interest rate is 6%? **(\$794.87)**

What would the PV be if the \$100 payments occurred in Years 1 through 10 and the \$500 came at the end of Year 10? **(\$1,015.21)**

(Continued)

11. On some calculators, instead of entering $CF_5 = 500$, you enter $CF_3 = 500$, because this is the next cash flow *different* from 300.

What is the present value of the following uneven cash flow stream: \$0 at Time 0, \$100 at the end of Year 1 (or at Time 1), \$200 at the end of Year 2, \$0 at the end of Year 3, and \$400 at the end of Year 4—assuming the interest rate is 8%? **(\$558.07)**

Would a “typical” common stock provide cash flows more like an annuity or more like an uneven cash flow stream?

28-13 Future Value of an Uneven Cash Flow Stream

The future value of an uneven cash flow stream (sometimes called the **terminal, or horizon, value**) is found by compounding each payment to the end of the stream and then summing the future values:

$$\begin{aligned}
 FV &= CF_0(1 + I)^N + CF_1(1 + I)^{N-1} + CF_2(1 + I)^{N-2} + \dots + CF_{N-1}(1 + I) + CF_N \\
 &= \sum_{t=0}^N CF_t(1 + I)^{N-t}
 \end{aligned}
 \tag{28-11}$$

The future value of our illustrative uneven cash flow stream is \$1,791.15, as shown in Figure 28-8.

FIGURE 28-8 Future Value of an Irregular Cash Flow Stream

	A	B	C	D	E	F	G	
498	INPUTS:							
499	Interest rate = I =		12%					
500								
501	1. Step-by-step:							
502	Periods:	0	1	2	3	4	5	
503	Cash flows:	\$0.00	\$100.00	\$300.00	\$300.00	\$300.00	\$500.00	
504	FVs of the CFs:		\$157.35	\$421.48	\$376.32	\$336.00	\$500.00	
505								
506	FV of the Irregular CF Stream = Sum of the Individual FVs =					\$1,791.15		
507								
508								
509	2. Calculator:	You could enter the cash flows into the cash flow register of a financial calculator, enter I/YR, and then press the NFV key to find the answer.					\$1,791.15	
510								
511	3. Excel Spreadsheet	Step 1. Find NPV:			=NPV(C499,C503:G503)		\$1,016.35	
512		Step 2. Compound NPV to find NFV:			=FV(C499,G502,0,-G511)		\$1,791.15	

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Some financial calculators have a net future value (NFV) key, which, after the cash flows and interest rate have been entered, can be used to obtain the future value of an uneven cash flow stream. If your calculator doesn't have the NFV feature, you can first find the net present value of the stream, and then find its net future value as $NFV = NPV(1 + I)^N$. In the illustrative problem, we find $PV = 1,016.35$ using the cash flow register and $I = 12$. Then we use the TVM register, entering $N = 5$, $I = 12$, $PV = -1016.35$, and $PMT = 0$. When we press FV , we find $FV = 1,791.15$, which is the same as the value shown on the time line in Figure 28-8. As Figure 28-8 also shows, the same procedure can be used with *Excel*.

SELF TEST

What is the future value of this cash flow stream: \$100 at the end of 1 year, \$150 after 2 years, and \$300 after 3 years, assuming the appropriate interest rate is 15%? (**\$604.75**)

28-14 Solving for I with Irregular Cash Flows

Before financial calculators and spreadsheets existed, it was *extremely difficult* to find I if the cash flows were uneven. However, with spreadsheets and financial calculators, it's easy to find I . If you have an *annuity plus a final lump sum*, you can input values for N , PV , PMT , and FV into the calculator's TVM registers and then press the I/YR key. Here's the setup for Stream 1 from Section 28-12, assuming we must pay \$927.90 to buy the asset:

Inputs	5	-927.90	100	1000
	N	PV	PMT	FV
Output		12.00		

The rate of return on the \$927.90 investment is 12%.

Finding the interest rate for an *irregular cash flow stream* with a calculator is a bit more complicated. Figure 28-9 shows Stream 2 from Section 28-12, assuming a required investment of $CF_0 = -\$1,000$. First, note that there is no simple step-by-step method for finding the rate of return—finding the rate for this investment requires a trial-and-error process, which is terribly time-consuming. Therefore, we really need a financial calculator or a spreadsheet. With a calculator, we would enter the CFs into the cash flow register and then press the IRR key to get the answer. IRR stands for “internal rate of return,” and it is the rate of return the investment provides. The investment is the cash flow at Time 0, and it must be entered as a negative number. When we enter those cash flows in the calculator's cash flow register and press the IRR key, we get the rate of return on the \$1,000 investment, 12.55%. Finally, note that once you have entered the cash flows in the calculator's register, you can find both the investment's net present value (NPV) and its internal rate of return. For investment decisions, we typically want both of these numbers. Therefore, we generally enter the data once and then find both the NPV and the IRR.

FIGURE 28-9 IRR of an Uneven Cash Flow Stream

	A	B	C	D	E	F	G
545	Periods:	0	1	2	3	4	5
546	Cash flows:	-\$1,000	\$100	\$300	\$300	\$300	\$500
547							
548	1. Calculator:	You could enter the cash flows into the cash flow register of a financial calculator and then press the IRR key to find the answer.					12.55%
549							
550	2. Excel IRR Function:	Cell references:	IRR =	=IRR(B546:G546)			12.55%

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You would get the same answer using *Excel's* IRR function, as shown in Figure 28-9. Notice that when using the IRR—unlike using the NPV function—you must include all cash flows, including the Time-0 cash flow.

An investment costs \$465 now and is expected to produce cash flows of \$100 at the end of each of the next 4 years, plus an extra lump-sum payment of \$200 at the end of the fourth year. What is the expected rate of return on this investment? **(9.05%)**

An investment costs \$465 and is expected to produce cash flows of \$100 at the end of Year 1, \$200 at the end of Year 2, and \$300 at the end of Year 3. What is the expected rate of return on this investment? **(11.71%)**

SELF TEST

28-15 Semiannual and Other Compounding Periods

In most of our examples thus far, we assumed that interest is compounded once a year, or annually. This is **annual compounding**. Suppose, however, that you put \$1,000 into a bank that pays a 6% annual interest rate but credits interest every 6 months. This is **semiannual compounding**. If you leave your funds in the account, how much would you have at the end of 1 year under semiannual compounding? Note that you will receive \$60 of interest for the year with annual compounding. With semiannual compounding, you will receive \$30 of it after only 6 months and you will earn interest on the first \$30 during the second 6 months, so you will end the year with more than the \$60 you would have had under annual compounding. You would be even better off under quarterly, monthly, weekly, or daily compounding. Note also that virtually all bonds pay interest semiannually; most stocks pay dividends quarterly; most mortgages, student loans, and auto loans involve monthly payments; and most money fund accounts pay interest daily. Therefore, it is essential that you understand how to deal with nonannual compounding.

28-15a Types of Interest Rates

When we move beyond annual compounding, we must deal with the following four types of interest rates:

- Nominal annual rates, given the symbol I_{NOM}
- Annual percentage rates, termed APR rates
- Periodic rates, denoted as I_{PER}
- Effective annual rates, given the symbol EAR or EFF%

Nominal (or Quoted) Rate, I_{NOM} ¹²

This is the rate quoted by banks, brokers, and other financial institutions. So, if you talk with a banker, broker, mortgage lender, auto finance company, or student loan officer about rates, the nominal rate is the one he or she will normally quote you. However, to be meaningful, the quoted nominal rate must also include the number of compounding periods per year. For example, a bank might offer you a CD at 6% compounded daily, while a credit union might offer 6.1% compounded monthly.

Note that the nominal rate is never shown on a time line, and it is never used as an input in a financial calculator (except when compounding occurs only once a year). If more frequent compounding occurs, you must use periodic rates.

Periodic Rate, I_{PER}

This is the rate charged by a lender or paid by a borrower each period. It can be a rate per year, per 6 months (semiannually), per quarter, per month, per day, or per any other time interval. For example, a bank might charge 1.5% per month on its credit card loans, or a finance company might charge 3% per quarter on installment loans.

We find the periodic rate as follows:

(28-12)

$$\text{Periodic rate } I_{\text{PER}} = I_{\text{NOM}}/M$$

where I_{NOM} is the nominal annual rate and M is the number of compounding periods per year. Thus, a 6% nominal rate with semiannual payments results in a periodic rate of

$$\text{Periodic rate } I_{\text{PER}} = 6\%/2 = 3.00\%.$$

If only one payment is made per year then $M = 1$, in which case the periodic rate would equal the nominal rate: $6\%/1 = 6\%$.

*The periodic rate is the rate shown on time lines and used in calculations.*¹³

To illustrate, suppose you invest \$100 in an account that pays a nominal rate

12. The term *nominal rate* as used here does not have the same meaning as it did in Chapter 1. There, nominal interest rates referred to stated market rates as opposed to real (zero-inflation) rates. In this chapter, the term *nominal rate* means the stated, or quoted, annual rate as opposed to the effective annual rate, which we explain later. In both cases, though, *nominal* means *stated*, or *quoted*, as opposed to some sort of adjusted rate.

13. The only exception is in cases where: (1) annuities are involved *and* (2) the payment periods do not correspond to the compounding periods. In such cases—for example, if you are making quarterly payments into a bank account to build up a specified future sum but the bank pays interest on a daily basis—then the calculations are more complicated. For such problems, the simplest procedure is to determine the periodic (daily) interest rate by dividing the nominal rate by 365 (or by 360 if the bank uses a 360-day year), then compound each payment over the exact number of days from the payment date to the terminal point, and then sum the compounded payments to find the future value of the annuity. This is a simple process with a computer.

of 12%, compounded quarterly, or 3% per period. How much would you have after 2 years if you leave the funds on deposit? First, here is the time line for the problem:



To find the FV, we would use this modified version of Equation 28-1:

$$FV_N = PV(1 + I_{\text{PER}})^{\text{Number of periods}} = PV\left(1 + \frac{I_{\text{NOM}}}{M}\right)^{MN} \quad (28-13)$$

$$= \$100\left(1 + \frac{0.12}{4}\right)^{4 \times 2} = \$100(1 + 0.03)^8 = \$126.68$$

With a financial calculator, we find the FV using these inputs: $N = 4 \times 2 = 8$, $I = 12/4 = 3$, $PV = -100$, and $PMT = 0$. The result is again $FV = \$126.68$:¹⁴

Inputs	8	3	-100	0	
	N	I/YR	PV	PMT	FV
Output					126.68

Effective (or Equivalent) Annual Rate (EAR or EFF%)

This is the annual (interest once a year) rate that produces the same final result as compounding at the periodic rate for M times per year. The EAR, also called EFF% (for effective percentage rate), is found as follows:¹⁵

$$\begin{aligned} \text{EAR} = \text{EFF}\% &= (1 + I_{\text{PER}})^M - 1.0 \\ &= \left(1 + \frac{I_{\text{NOM}}}{M}\right)^M - 1.0 \end{aligned} \quad (28-14)$$

14. Most financial calculators have a feature that allows you to set the number of payments per year and then use the nominal annual interest rate. However, students tend to make fewer errors when using the periodic rate with their calculators set for one payment per year (i.e., per period), so this is what we recommend. Note also that you cannot use a normal time unless you use the periodic rate.
15. You could also use the "interest conversion feature" of a financial calculator. Most financial calculators are programmed to find the EFF% or, given the EFF%, to find the nominal rate; this is called "interest rate conversion." You enter the nominal rate and the number of compounding periods per year, and then press the EFF% key to find the effective annual rate. However, we generally use Equation 28-14 because it's easy and because using the equation reminds us of what we are really doing. If you do use the interest rate conversion feature on your calculator, don't forget to reset your settings afterward. Interest conversion is discussed in our calculator tutorials.

Here I_{NOM}/M is the periodic rate, and M is the number of periods per year. If a bank would lend you money at a nominal rate of 12%, compounded quarterly, then the EFF% rate would be 12.5509%:

$$\begin{aligned}\text{Rate on bank loan: EFF\%} &= (1 + 0.03)^4 - 1.0 = (1.03)^4 - 1.0 \\ &= 1.125509 - 1.0 = 0.125509 = 12.5509\%\end{aligned}$$

To see the importance of the EFF%, suppose that—as an alternative to the bank loan—you could borrow on a credit card that charges 1% per month. Would you be better off using the bank loan or credit card loan? *To answer this question, the cost of each alternative must be expressed as an EFF%. We just saw that the bank loan's effective cost is 12.5509%. The cost of the credit card loan, with monthly payments, is slightly higher, 12.6825%:*

$$\begin{aligned}\text{Credit card loan: EFF\%} &= (1 + 0.01)^{12} - 1.0 = (1.01)^{12} - 1.0 \\ &= 1.126825 - 1.0 = 0.126825 = 12.6825\%\end{aligned}$$

This result is logical: Both loans have the same 12% nominal rate, yet you would have to make the first payment after only 1 month on the credit card versus 3 months under the bank loan.

The EFF% rate is rarely used in calculations. *However, it must be used to compare the effective costs of different loans or rates of return on different investments when payment periods differ, as in our example of the credit card versus a bank loan.*

28-15b The Result of Frequent Compounding

What would happen to the future value of an investment if interest were compounded annually, semiannually, quarterly, or some other less-than-annual period?

TRUTH IN LENDING: WHAT LOANS REALLY COST

Congress passed the Consumer Credit Protection Act in 1968. The Truth in Lending provisions in the Act require banks and other lenders to disclose the **annual percentage rate (APR)** they are charging. For example, suppose you plan to buy a fancy TV set that costs \$3,000, and the store offers you credit for 1 year at an “add-on” quoted rate of 8%. Here we first find the total dollars of interest by multiplying the \$3,000 you are borrowing times 8%, resulting in \$240. Add the interest to the \$3,000 cost of the TV, for a total loan of \$3,240. Then divide the total loan by 12 to get the monthly payments: $\$3,240/12 = \270 per month, with the first payment made at the time of purchase. Therefore, we have a 12-month annuity due with payments of \$270. Is your cost really the 8% that you were quoted?

To find the APR, set your calculator to Begin Mode, then enter $N = 12$, $PV = 3000$, $PMT = -270$, and $FV = 0$. Then, when you press the I/YR key, you get the periodic rate, 1.4313%. You then multiply by 12 to get the APR, 17.1758%. You could also find the EFF%, which is 18.5945%. We show these calculations using both the calculator and *Excel*, along with a time line that helps us visualize what's happening, in the chapter's *Excel Tool Kit*.

The 17.1758% APR that the dealer is required to report is a much better indicator of the loan's cost than the 8% nominal rate, but it still does not reflect the true cost, which is the 18.5945% effective annual rate. Thus, buying the TV on time would really cost you 18.5945%. If you don't know what's happening when you buy on time or borrow, you may pay a lot more than you think!

FIGURE 28-10 Effect on \$100 of Compounding More Frequently Than Once a Year

	A	B	C	D	E	F	G
568	Frequency of Compounding	Nominal Annual Rate	Periods per Year (M) ^a	Periodic Interest Rate (I _{PER})	Effective Annual Rate (EFF%) ^b	Future Value ^c	Percentage Increase in FV
569	Annual	12%	1	12.0000%	12.0000%	\$112.00	
570	Semiannual	12%	2	6.0000%	12.3600%	\$112.36	0.32%
571	Quarterly	12%	4	3.0000%	12.5509%	\$112.55	0.17%
572	Monthly	12%	12	1.0000%	12.6825%	\$112.68	0.12%
573	Daily	12%	365	0.0329%	12.7475%	\$112.75	0.06%

Notes:

^aWe used 365 days per year in the calculations.

^bThe EFF% is calculated as $(1 + I_{PER})^M - 1$.

^cThe future value is calculated as $\$100(1 + \text{EFF}\%)$.

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Because interest will be earned on interest more often, you should expect higher future values the more frequently compounding occurs. Similarly, you should expect the effective annual rate to increase with more frequent compounding. As Figure 28-10 shows, these results do occur—the future value and the EFF% do increase as the frequency of compounding increases. Notice that the biggest increase in FV (and in EFF%) occurs when compounding goes from annual to semiannual, and notice also that moving from monthly to daily compounding has a relatively small impact. Although Figure 28-10 shows daily compounding as the smallest interval, it is possible to compound even more frequently. At the limit, compounding can occur continuously. This is explained in *Web Extension 28C* on the textbook's Web site.

Would you rather *invest* in an account that pays a 7% nominal rate with annual compounding or with monthly compounding? If you *borrowed* at a nominal rate of 7%, would you rather make annual or monthly payments? Why?

What is the *future value* of \$100 after 3 years if the appropriate interest rate is 8%, compounded annually? **(\$125.97)** Compounded monthly? **(\$127.02)**

What is the *present value* of \$100 due in 3 years if the appropriate interest rate is 8%, compounded annually? **(\$79.38)** Compounded monthly? **(\$78.73)**

Define the following terms: annual percentage rate (APR), effective annual rate (EFF%), and nominal interest rate (I_{NOM}).

A bank pays 5% with daily compounding on its savings accounts. Should it advertise the nominal or effective rate if it is seeking to attract new deposits?

(Continued)

SELF TEST

Credit card issuers must by law print their annual percentage rate on their monthly statements. A common APR is 18%, with interest paid monthly. What is the EFF% on such a loan? **(19.56%)**

Some years ago banks weren't required to reveal the rate they charged on credit cards. Then Congress passed a "truth in lending" law that required them to publish their APR rate. Is the APR rate really the most truthful rate, or would the EFF% be even more truthful?

28-16 Fractional Time Periods¹⁶

So far we have assumed that payments occur at either the beginning or the end of periods, but not *within* periods. However, we occasionally encounter situations that require compounding or discounting over fractional periods. For example, suppose you deposited \$100 in a bank that pays a nominal rate of 10%, compounded daily, based on a 365-day year. How much would you have after 9 months? The answer of \$107.79 is found as follows:¹⁷

$$\text{Periodic rate} = I_{\text{PER}} = 0.10/365 = 0.000273973 \text{ per day}$$

$$\begin{aligned} \text{Number of days} &= (9/12)(365) = 0.75(365) \\ &= 273.75 \text{ days, rounded to } 274 \end{aligned}$$

$$\text{Ending amount} = \$100(1.000273973)^{274} = \$107.79$$

Now suppose that instead you borrow \$100 at a nominal rate of 10% per year and are charged *simple interest*, which means that interest is not charged on interest. If the loan is outstanding for 274 days (or 9 months), how much interest would you have to pay? The interest owed is equal to the principal multiplied by the interest rate times the number of periods. In this case, the number of periods is equal to a fraction of a year: $N = 274/365 = 0.7506849$.

$$\text{Interest owed} = \$100(10\%)(0.7506849) = \$7.51$$

Another approach would be to use the daily rate rather than the annual rate and thus to use the exact number of days rather than the fraction of the year:

$$\text{Interest owed} = \$100(0.000273973)(274) = \$7.51$$

You would owe the bank a total of \$107.51 after 274 days. This is the procedure most banks use to calculate interest on loans, except that they generally require

16. This section is interesting and useful but relatively technical. It can be omitted, at the option of the instructor, without loss of continuity.

17. We assume that these 9 months constitute 9/12 of a year. Also, bank deposit and loan contracts specifically state whether they are based on a 360-day or a 365-day year. If a 360-day year is used, then the daily rate is higher, so the effective rate is also higher. Here we assumed a 365-day year. Finally, note that banks use software with built-in calendars, so they calculate the exact number of days.

Note also that banks often treat such loans as follows: (1) They require monthly payments, and they calculate the interest for the month by multiplying the periodic rate by the beginning-of-month balance times the number of days in the month. This is called "simple interest." (2) The interest for the month is either added to the next beginning of month balance, or the borrower must actually pay the earned interest. In this case, the EFF% is based on 12 compounding periods, not 365 as is assumed in our example.

WHAT YOU KNOW IS WHAT YOU GET: NOT IN PAYDAY LENDING

When money runs low toward the end of a month, many individuals turn to payday lenders. If a borrower's application is approved, the payday lender makes a short-term loan, which will be repaid on with the next paycheck. In fact, on the next payday the lender actually transfers the repayment from the borrower's bank account. This repayment consists of the amount borrowed plus a fee.

How costly are payday loans? The lender charges a fee of about \$15 to \$17 per \$100 borrowed. A typical loan is for about \$350, so the typical fee is about \$56. A typical borrower gets paid about every 2 weeks, so the loan is for a very short amount of time. With a big fee and a short time until repayment, the typical payday loan has an APR of over 400%.

How informed are borrowers? Two professors at the University of Chicago set out to answer this question. When loans are approved, borrowers receive a form to sign that shows the APR. However, subsequent telephone surveys of borrowers show that over 40% of borrowers thought their APR was around 15%; perhaps not coincidentally, these are similar numerals to the fee schedules that are posted prominently in the lender's office.

The professors then did an experiment (with the agreement of 77 payday loan stores) in which they provided more information than just the APR. One group of borrowers received information about the APR of the payday loan as compared to the APRs of other loans, such as car loans. A second group received information about the dollar cost of the payday loan as compared to the dollar cost of other loans, such as car loans. A third group received information about how long it takes most payday borrowers to repay their loans (which is longer than the next payday; borrowers tend to extend the loan for additional pay periods, accruing additional fees).

Compared to a control group with no additional information, the results show that some borrowers with additional information decided not to take the loan; other borrowers reduced the amount that they borrow. These findings suggest that better information helps borrowers make less costly decisions. The more you know, the less you get, at least when it comes to costly payday loans.

Source: Marianne Bertrand and Adair Morse, "Information Disclosure, Cognitive Biases, and Payday Borrowing," *Journal of Finance*, Vol. 66, No. 6, December 2011, pp. 1865–1893.

borrowers to pay the interest on a monthly basis rather than after 274 days; this more frequent compounding raises the EFF% and thus the total amount of interest paid.

Suppose a company borrowed \$1 million at a rate of 9%, using simple interest, with interest paid at the end of each month. The bank uses a 360-day year. How much interest would the firm have to pay in a 30-day month? **(\$7,500.00)** What would the interest be if the bank used a 365-day year? **(\$7,397.26)**

Suppose you deposited \$1,000 in a credit union that pays 7% with daily compounding and a 365-day year. What is the EFF%? **(7.250098%)** How much could you withdraw after 7 months, assuming this is 7/12 of a year? **(\$1,041.67)**

SELF TEST

28-17 Amortized Loans

An extremely important application of compound interest involves loans that are paid off in installments over time. Included are automobile loans, home mortgage loans, student loans, and many business loans. A loan that is to be repaid in equal amounts on a monthly, quarterly, or annual basis is called an **amortized loan**.¹⁸

28-17a Payments

Suppose a company borrows \$100,000, with the loan to be repaid in 5 equal payments at the end of each of the next 5 years. The lender charges 6% on the balance at the beginning of each year.

Here's a picture of the situation:



Our task is to find the amount of the payment, PMT, such that the sum of their PVs equals the amount of the loan, \$100,000:

$$\$100,000 = \frac{\text{PMT}}{(1.06)^1} + \frac{\text{PMT}}{(1.06)^2} + \frac{\text{PMT}}{(1.06)^3} + \frac{\text{PMT}}{(1.06)^4} + \frac{\text{PMT}}{(1.06)^5} = \sum_{t=1}^5 \frac{\text{PMT}}{(1.06)^t}$$

It is possible to solve the annuity formula, Equation 28-8, for PMT, but it is much easier to use a financial calculator or spreadsheet. With a financial calculator, we insert values as shown below to get the required payments, \$23,739.64:

Inputs	5	6	100000		0
	N	I/YR	PV	PMT	FV
Output				-23739.64	

With *Excel*, you would use the PMT function: `=PMT(I,N,PV,FV)` = `PMT(0.06,5,100000,0)` = -\$23,739.64. Thus, we see that the borrower must pay the lender \$23,739.64 per year for the next 5 years.

28-17b Amortization Schedules

Each payment will consist of two parts—part interest and part repayment of principal. This breakdown is shown in the **amortization schedule** given in Figure 28-11. The interest component is relatively high in the first year, but it declines as the loan balance decreases. For tax purposes, the borrower would deduct the interest component while the lender would report the same amount as taxable income. Over the 5 years, the lender will earn 6% on its investment and also recover the amount of its investment.

18. The word *amortized* comes from the Latin *mors*, meaning “death,” so an amortized loan is one that is “killed off” over time.

FIGURE 28-11 Loan Amortization Schedule, \$100,000 at 6% for 5 Years

	A	B	C	D	E	F
675	INPUTS:					
676	Amount borrowed:		\$100,000			
677	Years:		5			
678	Rate:		6%			
679	Intermediate calculation:					
680	PMT: \$23,739.64 =PMT(C678,C677,-C676)					
	Year	Beginning Amount (1)	Payment (2)	Interest ^a (3)	Repayment of Principal ^b (2) - (3) = (4)	Ending Balance (1) - (4) = (5)
681						
682	1	\$100,000.00	\$23,739.64	\$6,000.00	\$17,739.64	\$82,260.36
683	2	\$82,260.36	\$23,739.64	\$4,935.62	\$18,804.02	\$63,456.34
684	3	\$63,456.34	\$23,739.64	\$3,807.38	\$19,932.26	\$43,524.08
685	4	\$43,524.08	\$23,739.64	\$2,611.44	\$21,128.20	\$22,395.89
686	5	\$22,395.89	\$23,739.64	\$1,343.75	\$22,395.89	\$0.00

Notes:

^aInterest in each period is calculated by multiplying the loan balance at the beginning of the year by the interest rate. Therefore, interest in Year 1 is \$100,000(0.06) = \$6,000; in Year 2 it is \$82,260.36(0.06) = \$4,935.62; and so on.

^bRepayment of principal is the \$23,739.64 annual payment minus the interest charge for the year, \$17,739.64 for Year 1.

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28-17c Mortgage Interest Payments

Now consider a 30-year home mortgage of \$250,000 at an annual rate of 6%. How much interest will the borrower pay over the life of the loan? How much in the first year?

Begin by finding the monthly payment. The financial calculator inputs are shown below (notice that N and I/YR are adjusted to reflect monthly payments):

Inputs	360	0.5%	250000	0
	N	I/YR	PV	FV
Output			-1498.8763	

The total amount of payments is 360(\$1,498.8763) ≈ \$539,595. The borrower pays back the borrowed \$250,000 over the life of the loan, so the total interest paid is \$539,595 - \$250,000 = \$289,595.

To find the amount of interest paid in the first year, we could construct a monthly amortization schedule as in Figure 28-11 and add up the monthly interest payments for the first year. There is an easier way, though. Begin by finding the amount the borrower owes at the end of the first year. We know the number of

THE GLOBAL ECONOMIC CRISIS

An Accident Waiting to Happen: Option Reset Adjustable Rate Mortgages

Option reset adjustable rate mortgages (ARMs) give the borrower some choices regarding the initial monthly payment. One popular option ARM allowed borrowers to make a monthly payment equal to only half of the interest due in the first month. Because the monthly payment was less than the interest charge, the loan balance grew each month. When the loan balance exceeded 110% of the original principal, the monthly payment was reset to fully amortize the now-larger loan at the prevailing market interest rates.

Here's an example. Someone borrows \$325,000 for 30 years at an initial rate of 7%. The interest accruing in the first month is $(7\%/12)(\$325,000) = \$1,895.83$. Therefore, the initial monthly payment is $50\%(\$1,895.83) = \947.92 . Another \$947.92 of deferred interest is added to the loan balance, taking it up to $\$325,000 + \$947.92 = \$325,947.82$. Because the loan is now larger, interest in the second month is higher, and both interest and the loan balance will continue to rise each month. The first month after the loan balance exceeds $110\%(\$325,000) = \$357,500$, the contract calls for the payment to be reset so as to fully amortize the loan at the then-prevailing interest rate.

First, how long would it take for the balance to exceed \$357,500? Consider this from the lender's perspective: The lender initially pays out \$325,000, receives \$947.92 each month, and then would receive a payment of \$357,500 if the loan were payable when the balance

hit that amount, with interest accruing at a 7% annual rate and with monthly compounding. We enter these values into a financial calculator: $I = 7\%/12$, $PV = -325000$, $PMT = 947.92$, and $FV = 357500$. We solve for $N = 31.3$ months, rounded up to 32 months. Thus, the borrower will make 32 payments of \$947.92 before the ARM resets.

The payment after the reset depends upon the terms of the original loan and the market interest rate at the time of the reset. For many borrowers, the initial rate was a lower-than-market "teaser" rate, so a higher-than-market rate would be applied to the remaining balance. For this example, we will assume that the original rate wasn't a teaser and that the rate remains at 7%. Keep in mind, though, that for many borrowers the reset rate was higher than the initial rate. The balance after the 32nd payment can be found as the future value of the original loan and the 32 monthly payments, so we enter these values in the financial calculator: $N = 32$, $I = 7\%/12$, $PMT = 947.92$, $PV = -325000$, and then solve for $FV = \$358,242.84$. The number of remaining payments to amortize the \$358,424.84 loan balance is $360 - 32 = 328$, so the amount of each payment is found by setting up the calculator as $N = 328$, $I = 7\%/12$, $PV = 358242.84$, and $FV = 0$. Solving, we find that $PMT = \$2,453.94$.

Even if interest rates don't change, the monthly payment jumps from \$947.92 to \$2,453.94 and would increase even more if interest rates were higher at the reset. This is exactly what happened to millions of American homeowners who took out option reset ARMs in the early 2000s. When large numbers of resets began in 2007, defaults ballooned. The accident caused by option reset ARMs didn't wait very long to happen!

remaining payments ($360 - 12 = 348$) and the amount of each payment (\$1,498.88), so we can solve for the PV:

Inputs	348	0.5%	-1498.88	0
	N	I/YR	PV	PMT
Output	246930.58			

The amount of principal repaid in the first year is the difference between the starting value and the value at the end of the year: $\$250,000 - \$246,930.58 =$

\$3,069.42. The total payments during the year are $12(\$1,498.88) = \$17,986.56$. So the total interest paid in the year is $\$17,986.56 - \$3,069.42 = \$14,917.14$. Almost 83% of the payments in the first year go to interest!

Now consider a 15-year mortgage. To compare apples to apples, assume the interest rate stays at 6%, although it probably would be a bit lower. Changing N to 180, the new payment is $\$2,109.6421$. The total amount of payments is $180(\$2,109.6421) \approx \$379,736$ and the total interest paid is $\$379,736 - \$250,000 = \$129,736$, a big decrease from the $\$289,595$ paid on the 30-year mortgage. As this example shows, increasing the monthly payment can dramatically reduce the total interest paid and the time required to pay off the mortgage.

Consider again the example in Figure 28-11. If the loan were amortized over 5 years with 60 equal monthly payments, how much would each payment be, and how would the first payment be divided between interest and principal? **(Each payment would be \$1,933.28; the first payment would have \$500 of interest and \$1,433.28 of principal repayment.)**

Suppose you borrowed \$30,000 on a student loan at a rate of 8% and now must repay it in three equal installments at the end of each of the next 3 years. How large would your payments be, how much of the first payment would represent interest and how much would be principal, and what would your ending balance be after the first year? **(PMT = \$11,641.01; interest = \$2,400; principal = \$9,241.01; balance at end of Year 1 = \$20,758.99)**

SELF TEST

28-18 Growing Annuities¹⁹

Normally, an annuity is defined as a series of *constant* payments to be received over a specified number of periods. However, the term **growing annuity** is used to describe a series of payments that grow at a constant rate.

28-18a Example 1: Finding a Constant Real Income

Growing annuities are often used in the area of financial planning, where a prospective retiree wants to determine the maximum constant *real*, or *inflation-adjusted*, withdrawals that he or she can make over a specified number of years. For example, suppose a 65-year-old is contemplating retirement. The individual expects to live for another 20 years, has a \$1 million nest egg, expects the investments to earn a nominal annual rate of 6%, expects inflation to average 3% per year, and wants to withdraw a constant *real* amount annually over the next 20 years so as to maintain a constant standard of living. If the first withdrawal is to be made today, what is the amount of that initial withdrawal?

19. This section is interesting and useful but relatively technical. It can be omitted, at the option of the instructor, without loss of continuity.

This problem can be solved in three ways: (1) Set up a spreadsheet model that is similar to an amortization table, where the account earns 6% per year, withdrawals rise at the 3% inflation rate, and Excel's Goal Seek function is used to find the initial inflation-adjusted withdrawal. A zero balance will be shown at the end of the twentieth year. (2) Use a financial calculator, where we first calculate the real rate of return, adjusted for inflation, and use it for I/YR when finding the payment for an annuity due. (3) Use a relatively complicated and obtuse formula to find this same amount.²⁰ We will focus on the first two approaches.

WEB

See *Ch28 Tool Kit.xls* for all calculations.

We illustrate the spreadsheet approach in the chapter model, *Ch28 Tool Kit.xls*. The spreadsheet model provides the most transparent picture, because it shows the value of the retirement portfolio, the portfolio's annual earnings, and each withdrawal over the 20-year planning horizon—especially if you include a graph. A picture is worth a thousand numbers, and graphs make it easy to explain the situation to people who are planning their financial futures.

To implement the calculator approach, we first find the expected *real* rate of return, where r_r is the real rate of return and r_{NOM} the nominal rate of return. The real rate of return is the return that we would see if there were no inflation. We calculate the real rate as:

(28-15)

$$\text{Real rate} = r_r = [(1 + r_{\text{NOM}})/(1 + \text{Inflation})] - 1.0$$

$$= [1.06/1.03] - 1.0 = 0.029126214 = 2.9126214\%$$

Using this real rate of return, we solve the annuity due problem exactly as we did earlier in the chapter. We set the calculator to Begin Mode, after which we input $N = 20$, $I/YR = \text{real rate} = 2.9126214$, $PV = -1000000$, and $FV = 0$; then we press PMT to get \$64,786.88. This is the amount of the initial withdrawal at Time 0 (today), and future withdrawals will increase at the inflation rate of 3%. These withdrawals, growing at the inflation rate, will provide the retiree with a constant real income over the next 20 years—provided the inflation rate and the rate of return do not change.

In our example, we assumed that the first withdrawal would be made immediately. The procedure would be slightly different if we wanted to make end-of-year withdrawals. First, we would set the calculator to End Mode. Second, we would enter the same inputs into the calculator as just listed, including the real interest rate for I/YR. The calculated PMT would be \$66,673.87. *However, that value is in beginning-of-year terms*, and because inflation of 3% will occur during the

20. For example, the formula used to find the payment of a growing annuity due is shown below. If g = annuity growth rate and r = nominal rate of return on investment, then:

$$\text{PVIF of a growing annuity due} = \text{PVIFGA}_{\text{due}} = \{1 - [(1 + g)/(1 + r)]^N\} [(1 + r)/(r - g)]$$

$$\text{PMT} = \text{PV}/\text{PVIFGA}_{\text{due}}$$

where PVIF denotes "present value interest factor." Similar formulas are available for growing ordinary annuities.

year, we must make the following adjustment to find the inflation-adjusted initial withdrawal:

$$\begin{aligned}\text{Initial end-of-year withdrawal} &= \$66,673.87(1 + \text{Inflation}) \\ &= \$66,673.87(1.03) \\ &= \$68,674.09\end{aligned}$$

Thus, the first withdrawal at the *end* of the year would be \$68,674.09; it would grow by 3% per year; and after the 20th withdrawal (at the end of the 20th year), the balance in the retirement fund would be zero.

We also demonstrate the solution for this end-of-year payment example in *Ch28 Tool Kit.xls*. There we set up a table showing the beginning balance, the annual withdrawals, the annual earnings, and the ending balance for each of the 20 years. This analysis confirms the \$68,674.09 initial end-of-year withdrawal derived previously.

28-18b Example 2: Initial Deposit to Accumulate a Future Sum

As another example of growing annuities, suppose you need to accumulate \$100,000 in 10 years. You plan to make a deposit in a bank now, at Time 0, and then make 9 more deposits at the beginning of each of the following 9 years, for a total of 10 deposits. The bank pays 6% interest, you expect inflation to be 2% per year, and you plan to increase your annual deposits at the inflation rate. How much must you deposit initially? First, we calculate the real rate:

$$\text{Real rate} = r_r = [1.06/1.02] - 1.0 = 0.0392157 = 3.9215686\%$$

Next, because inflation is expected to be 2% per year, in 10 years the target \$100,000 will have a real value of:

$$\$100,000/(1 + 0.02)^{10} = \$82,034.83$$

Now we can find the size of the required initial payment by setting a financial calculator to the Begin Mode and then inputting $N = 10$, $I/YR = 3.9215686$, $PV = 0$, and $FV = 82034.83$. Then, when we press the PMT key, we get $PMT = -6,598.87$. Thus, a deposit of \$6,598.87 made at time 0 and growing by 2% per year will accumulate to \$100,000 by Year 10 if the interest rate is 6%. Again, this result is confirmed in the chapter's *Tool Kit*. The key to this analysis is to express I/YR, FV, and PMT in real, not nominal, terms.

Differentiate between a “regular” and a “growing” annuity.

What three methods can be used to deal with growing annuities?

If the nominal interest rate is 10% and the expected inflation rate is 5%, what is the expected real rate of return? **(4.7619%)**

SELF TEST

Summary

Most financial decisions involve situations in which someone makes a payment at one point in time and receives money later. Dollars paid or received at two different points in time are different, and this difference is dealt with using *time value of money (TVM) analysis*.

- **Compounding** is the process of determining the **future value (FV)** of a cash flow or a series of cash flows. The compounded amount, or future value, is equal to the beginning amount plus interest earned.
- Future value of a single payment = $FV_N = PV(1 + I)^N$.
- **Discounting** is the process of finding the **present value (PV)** of a future cash flow or a series of cash flows; discounting is the reciprocal, or reverse, of compounding.

- Present value of a payment received at the end of Time N = $PV = \frac{FV_N}{(1 + I)^N}$.

- An **annuity** is defined as a series of equal periodic payments (PMT) for a specified number of periods.
- An annuity whose payments occur at the *end* of each period is called an **ordinary annuity**.

- Future value of an (ordinary) annuity = $FVA_N = PMT \left[\frac{(1 + I)^N}{I} - \frac{1}{I} \right]$.

- Present value of an (ordinary) annuity = $PVA_N = PMT \left[\frac{1}{I} - \frac{1}{I(1 + I)^N} \right]$.

- If payments occur at the *beginning* of the periods rather than at the end, then we have an **annuity due**. The PV of each payment is larger, because each payment is discounted back 1 year less, so the PV of the annuity is also larger. Similarly, the FV of the annuity due is larger because each payment is compounded for an extra year. The following formulas can be used to convert the PV and FV of an ordinary annuity to an annuity due:

$$PVA_{\text{due}} = PVA_{\text{ordinary}} (1 + I)$$

$$FVA_{\text{due}} = FVA_{\text{ordinary}} (1 + I)$$

- A **perpetuity** is an annuity with an infinite number of payments:

$$\text{Value of a perpetuity} = \frac{PMT}{I}$$

- To find the PV or FV of an uneven series, find the PV or FV of each individual cash flow and then sum them.
- If you know the cash flows and the PV (or FV) of a cash flow stream, you can **determine its interest rate**.
- When compounding occurs more frequently than once a year, the nominal rate must be converted to a periodic rate, and the number of years must be converted to periods:

$$\text{Periodic rate } (I_{\text{PER}}) = \text{Nominal annual rate} \div \text{Periods per year}$$

$$\text{Number of Periods} = \text{Years} \times \text{Periods per year}$$

The periodic rate and number of periods is used for calculations and is shown on time lines.

- If you are comparing the costs of alternative loans that require payments more than once a year, or the rates of return on investments that pay interest more than once a year, then the comparisons should be based on **effective** (or **equivalent**) **rates** of return. Here is the formula:

$$\text{EAR} = \text{EFF}\% = (1 + I_{\text{PER}})^M - 1.0 = \left(1 + \frac{I_{\text{NOM}}}{M}\right)^M - 1.0$$

- The general equation for finding the future value of a current cash flow (PV) for any number of compounding periods per year is:

$$\text{FV}_N = \text{PV}(1 + I_{\text{PER}})^{\text{Number of periods}} = P\left(1 + \frac{I_{\text{NOM}}}{M}\right)^{MN}$$

where

I_{NOM} = Nominal quoted interest rate

M = Number of compounding periods per year

N = Number of years

- An **amortized loan** is paid off with equal payments over a specified period. An **amortization schedule** shows how much of each payment constitutes interest, how much is used to reduce the principal, and the unpaid balance at the end of each period. The unpaid balance at Time N must be zero.
- A **growing annuity** is a stream of cash flows that grows at a constant rate for a specified number of years. The present and future values of growing annuities can be found with relatively complicated formulas or, more easily, with an *Excel* model.
- *Web Extension 28A* explains the **tabular approach**.
- *Web Extension 28B* provides derivations of the annuity formulas.
- *Web Extension 28C* explains **continuous compounding**.

Questions

28-1 Define each of the following terms:

- PV; I; INT; FV_N ; PVA_N ; FVA_N ; PMT; M ; I_{NOM}
- Opportunity cost rate
- Annuity; lump-sum payment; cash flow; uneven cash flow stream
- Ordinary (or deferred) annuity; annuity due
- Perpetuity; consol
- Outflow; inflow; time line; terminal value
- Compounding; discounting
- Annual, semiannual, quarterly, monthly, and daily compounding
- Effective annual rate (EAR or EFF%); nominal (quoted) interest rate; APR; periodic rate
- Amortization schedule; principal versus interest component of a payment; amortized loan

28-2 What is an *opportunity cost rate*? How is this rate used in discounted cash flow analysis, and where is it shown on a time line? Is the opportunity rate a single number that is used to evaluate all potential investments?

- 28-3 An *annuity* is defined as a series of payments of a fixed amount for a specific number of periods. Thus, \$100 a year for 10 years is an annuity, but \$100 in Year 1, \$200 in Year 2, and \$400 in Years 3 through 10 does *not* constitute an annuity. However, the entire series *does contain* an annuity. Is this statement true or false?
- 28-4 If a firm's earnings per share grew from \$1 to \$2 over a 10-year period, the *total growth* would be 100%, but the *annual growth rate* would be *less than* 10%. True or false? Explain.
- 28-5 Would you rather have a savings account that pays 5% interest compounded semiannually or one that pays 5% interest compounded daily? Explain.

Problems

Answers Appear in Appendix B

Easy Problems 1-8

- 28-1 **Future Value of a Single Payment**
If you deposit \$10,000 in a bank account that pays 10% interest annually, how much will be in your account after 5 years?
- 28-2 **Present Value of a Single Payment**
What is the present value of a security that will pay \$5,000 in 20 years if securities of equal risk pay 7% annually?
- 28-3 **Interest Rate on a Single Payment**
Your parents will retire in 18 years. They currently have \$250,000, and they think they will need \$1 million at retirement. What annual interest rate must they earn to reach their goal, assuming they don't save any additional funds?
- 28-4 **Number of Periods of a Single Payment**
If you deposit money today in an account that pays 6.5% annual interest, how long will it take to double your money?
- 28-5 **Number of Periods for an Annuity**
You have \$42,180.53 in a brokerage account, and you plan to deposit an additional \$5,000 at the end of every future year until your account totals \$250,000. You expect to earn 12% annually on the account. How many years will it take to reach your goal?
- 28-6 **Future Value: Ordinary Annuity versus Annuity Due**
What is the future value of a 7%, 5-year ordinary annuity that pays \$300 each year? If this were an annuity due, what would its future value be?
- 28-7 **Present and Future Value of an Uneven Cash Flow Stream**
An investment will pay \$100 at the end of each of the next 3 years, \$200 at the end of Year 4, \$300 at the end of Year 5, and \$500 at the end of Year 6. If other investments of equal risk earn 8% annually, what is this investment's present value? Its future value?

28-8 Annuity Payment and EAR

You want to buy a car, and a local bank will lend you \$20,000. The loan would be fully amortized over 5 years (60 months), and the nominal interest rate would be 12%, with interest paid monthly. What is the monthly loan payment? What is the loan's EFF%?

Intermediate Problems 9-29**28-9 Present and Future Values of Single Cash Flows for Different Periods**

Find the following values, *using the equations*, and then work the problems using a financial calculator to check your answers. Disregard rounding differences. (*Hint*: If you are using a financial calculator, you can enter the known values and then press the appropriate key to find the unknown variable. Then, without clearing the TVM register, you can “override” the variable that changes by simply entering a new value for it and then pressing the key for the unknown variable to obtain the second answer. This procedure can be used in parts b and d, and in many other situations, to see how changes in input variables affect the output variable.)

- An initial \$500 compounded for 1 year at 6%
- An initial \$500 compounded for 2 years at 6%
- The present value of \$500 due in 1 year at a discount rate of 6%
- The present value of \$500 due in 2 years at a discount rate of 6%

28-10 Present and Future Values of Single Cash Flows for Different Interest Rates

Use both the TVM equations and a financial calculator to find the following values. See the Hint for Problem 28-9.

- An initial \$500 compounded for 10 years at 6%
- An initial \$500 compounded for 10 years at 12%
- The present value of \$500 due in 10 years at a 6% discount rate
- The present value of \$500 due in 10 years at a 12% discount rate

28-11 Time for a Lump Sum to Double

To the closest year, how long will it take \$200 to double if it is deposited and earns the following rates? [*Notes*: (1) See the Hint for Problem 28-9. (2) This problem cannot be solved exactly with some financial calculators. For example, if you enter $PV = -200$, $PMT = 0$, $FV = 400$, and $I = 7$ in an HP-12C and then press the N key, you will get 11 years for part a. The correct answer is 10.2448 years, which rounds to 10, but the calculator rounds up. However, the HP10BII gives the exact answer.]

- 7%
- 10%
- 18%
- 100%

28-12 Future Value of an Annuity

Find the *future value* of the following annuities. The first payment in these annuities is made at the *end* of Year 1, so they are *ordinary annuities*. (*Notes*: See the Hint to Problem 28-9. Also, note that you can leave values

in the TVM register, switch to Begin Mode, press FV, and find the FV of the annuity due.)

- \$400 per year for 10 years at 10%
- \$200 per year for 5 years at 5%
- \$400 per year for 5 years at 0%
- Now rework parts a, b, and c assuming that payments are made at the *beginning* of each year; that is, they are *annuities due*.

28-13 Present Value of an Annuity

Find the *present value* of the following *ordinary annuities*. (see the *Notes* to Problem 28-12.)

- \$400 per year for 10 years at 10%
- \$200 per year for 5 years at 5%
- \$400 per year for 5 years at 0%
- Now rework parts a, b, and c assuming that payments are made at the *beginning* of each year; that is, they are *annuities due*.

28-14 Uneven Cash Flow Stream

- Find the present values of the following cash flow streams. The appropriate interest rate is 8%. (*Hint*: It is fairly easy to work this problem dealing with the individual cash flows. However, if you have a financial calculator, read the section of the manual that describes how to enter cash flows such as the ones in this problem. This will take a little time, but the investment will pay huge dividends throughout the course. Note that, when working with the calculator's cash flow register, you must enter $CF_0 = 0$. Note also that it is quite easy to work the problem with *Excel*, using procedures described in the file *Ch28 Tool Kit.xls*.)

Year	Cash Stream A	Cash Stream B
1	\$100	\$300
2	400	400
3	400	400
4	400	400
5	300	100

- What is the value of each cash flow stream at a 0% interest rate?

28-15 Effective Rate of Interest

Find the interest rate (or rates of return) in each of the following situations:

- You *borrow* \$700 and promise to pay back \$749 at the end of 1 year.
- You *lend* \$700 and receive a promise to be paid \$749 at the end of 1 year.
- You borrow \$85,000 and promise to pay back \$201,229 at the end of 10 years.
- You borrow \$9,000 and promise to make payments of \$2,684.80 at the end of each of the next 5 years.

28-16 Future Value for Various Compounding Periods

Find the amount to which \$500 will grow under each of the following conditions:

- a. 12% compounded annually for 5 years
- b. 12% compounded semiannually for 5 years
- c. 12% compounded quarterly for 5 years
- d. 12% compounded monthly for 5 years

28-17 Present Value for Various Compounding Periods

Find the present value of \$500 due in the future under each of the following conditions:

- a. 12% nominal rate, semiannual compounding, discounted back 5 years
- b. 12% nominal rate, quarterly compounding, discounted back 5 years
- c. 12% nominal rate, monthly compounding, discounted back 1 year

28-18 Future Value of an Annuity for Various Compounding Periods

Find the future values of the following ordinary annuities:

- a. FV of \$400 each 6 months for 5 years at a nominal rate of 12%, compounded semiannually
- b. FV of \$200 each 3 months for 5 years at a nominal rate of 12%, compounded quarterly
- c. The annuities described in parts a and b have the same total amount of money paid into them during the 5-year period, and both earn interest at the same nominal rate, yet the annuity in part b earns \$101.75 more than the one in part a over the 5 years. Why does this occur?

28-19 Effective versus Nominal Interest Rates

Universal Bank pays 7% interest, compounded annually, on time deposits. Regional Bank pays 6% interest, compounded quarterly.

- a. Based on effective interest rates, in which bank would you prefer to deposit your money?
- b. Could your choice of banks be influenced by the fact that you might want to withdraw your funds during the year as opposed to at the end of the year? In answering this question, assume that funds must be left on deposit during an entire compounding period in order for you to receive any interest.

28-20 Amortization Schedule

Consider a \$25,000 loan to be repaid in equal installments at the end of each of the next 5 years. The interest rate is 10%.

- a. Set up an amortization schedule for the loan.
- b. How large must each annual payment be if the loan is for \$50,000? Assume that the interest rate remains at 10% and that the loan is still paid off over 5 years.
- c. How large must each payment be if the loan is for \$50,000, the interest rate is 10%, and the loan is paid off in equal installments at the end of each of the next 10 years? This loan is for the same amount as the loan in part b, but the payments are spread out over twice as many periods. Why are these payments not half as large as the payments on the loan in part b?

28-21 Growth Rates

Sales for Hanebury Corporation's just-ended year were \$12 million. Sales were \$6 million 5 years earlier.

- a. At what rate did sales grow?
- b. Suppose someone calculated the sales growth for Hanebury in part a as follows: "Sales doubled in 5 years. This represents a growth of 100% in

5 years; dividing 100% by 5 results in an estimated growth rate of 20% per year.” Explain what is wrong with this calculation.

28–22 Expected Rate of Return

Washington-Pacific (W-P) invested \$4 million to buy a tract of land and plant some young pine trees. The trees can be harvested in 10 years, at which time W-P plans to sell the forest at an expected price of \$8 million. What is W-P’s expected rate of return?

28–23 Effective Rate of Interest

A mortgage company offers to lend you \$85,000; the loan calls for payments of \$8,273.59 at the end of each year for 30 years. What interest rate is the mortgage company charging you?

28–24 Required Lump-Sum Payment

To complete your last year in business school and then go through law school, you will need \$10,000 per year for 4 years, starting next year (that is, you will need to withdraw the first \$10,000 one year from today). Your uncle offers to put you through school, and he will deposit in a bank paying 7% interest a sum of money that is sufficient to provide the four payments of \$10,000 each. His deposit will be made today.

- How large must the deposit be?
- How much will be in the account immediately after you make the first withdrawal? After the last withdrawal?

28–25 Repaying a Loan

While Mary Corens was a student at the University of Tennessee, she borrowed \$12,000 in student loans at an annual interest rate of 9%. If Mary repays \$1,500 per year, then how long (to the nearest year) will it take her to repay the loan?

28–26 Reaching a Financial Goal

You need to accumulate \$10,000. To do so, you plan to make deposits of \$1,250 per year—with the first payment being made a year from today—into a bank account that pays 12% annual interest. Your last deposit will be less than \$1,250 if less is needed to round out to \$10,000. How many years will it take you to reach your \$10,000 goal, and how large will the last deposit be?

28–27 Present Value of a Perpetuity

What is the present value of a perpetuity of \$100 per year if the appropriate discount rate is 7%? If interest rates in general were to double and the appropriate discount rate rose to 14%, what would happen to the present value of the perpetuity?

28–28 PV and Effective Annual Rate

Assume that you inherited some money. A friend of yours is working as an unpaid intern at a local brokerage firm, and her boss is selling securities that call for 4 payments of \$50 (1 payment at the end of each of the next 4 years) plus an extra payment of \$1,000 at the end of Year 4. Your friend says she can get you some of these securities at a cost of \$900 each. Your money is now invested in a bank that pays an 8% nominal (quoted) interest rate but with quarterly compounding. You regard the securities as being just as safe,

and as liquid, as your bank deposit, so your required effective annual rate of return on the securities is the same as that on your bank deposit. You must calculate the value of the securities to decide whether they are a good investment. What is their present value to you?

28–29 Loan Amortization

Assume that your aunt sold her house on December 31, and to help close the sale she took a second mortgage in the amount of \$10,000 as part of the payment. The mortgage has a quoted (or nominal) interest rate of 10%; it calls for payments every 6 months, beginning on June 30, and is to be amortized over 10 years. Now, 1 year later, your aunt must inform the IRS and the person who bought the house about the interest that was included in the two payments made during the year. (This interest will be income to your aunt and a deduction to the buyer of the house.) To the closest dollar, what is the total amount of interest that was paid during the first year?

Challenging Problems 30–34

28–30 Loan Amortization

Your company is planning to borrow \$1 million on a 5-year, 15%, annual payment, fully amortized term loan. What fraction of the payment made at the end of the second year will represent repayment of principal?

28–31 Nonannual Compounding

It is now January 1. You plan to make a total of 5 deposits of \$100 each, one every 6 months, with the first payment being made *today*. The bank pays a nominal interest rate of 12% but uses *semiannual* compounding. You plan to leave the money in the bank for 10 years.

- How much will be in your account after 10 years?
- You must make a payment of \$1,432.02 in 10 years. To get the money for this payment, you will make five equal deposits, beginning today and for the following 4 quarters, in a bank that pays a nominal interest rate of 12% with *quarterly compounding*. How large must each of the five payments be?

28–32 Nominal Rate of Return

Anne Lockwood, manager of Oaks Mall Jewelry, wants to sell on credit, giving customers 3 months to pay. However, Anne will have to borrow from her bank to carry the accounts receivable. The bank will charge a nominal rate of 15% and will compound monthly. Anne wants to quote a nominal rate to her customers (all of whom are expected to pay on time) that will exactly offset her financing costs. What nominal annual rate should she quote to her credit customers?

28–33 Required Annuity Payments

Assume that your father is now 50 years old, plans to retire in 10 years, and expects to live for 25 years after he retires—that is, until age 85. He wants his first retirement payment to have the same purchasing power at the time he retires as \$40,000 has today. He wants all of his subsequent retirement payments to be equal to his first retirement payment. (Do not let the retirement payments grow with inflation: Your father realizes that if inflation occurs the real value of his retirement income will decline year by year after

he retires.) His retirement income will begin the day he retires, 10 years from today, and he will then receive 24 additional annual payments. Inflation is expected to be 5% per year from today forward. He currently has \$100,000 saved and expects to earn a return on his savings of 8% per year with annual compounding. To the nearest dollar, how much must he save during each of the next 10 years (with equal deposits being made at the end of each year, beginning a year from today) to meet his retirement goal? (*Note:* Neither the amount he saves nor the amount he withdraws upon retirement is a growing annuity.)

28–34 Growing Annuity Payments

You want to accumulate \$1 million by your retirement date, which is 25 years from now. You will make 25 deposits in your bank, with the first occurring *today*. The bank pays 8% interest, compounded annually. You expect to receive annual raises of 3%, which will offset inflation, and you will let the amount you deposit each year also grow by 3% (i.e., your second deposit will be 3% greater than your first, the third will be 3% greater than the second, etc.). How much must your first deposit be if you are to meet your goal?

Spreadsheet Problem

WEB

28–35 Build a Model: The Time Value of Money

Start with the partial model in the file *Ch28 P35 Build a Model.xls* from the textbook's Web site. Answer the following questions, using the spreadsheet model to do the calculations.

- Find the FV of \$1,000 invested to earn 10% annually 5 years from now. Answer this question first by using a math formula and then by using the *Excel* function wizard.
- Now create a table that shows the FV at 0%, 5%, and 20% for 0, 1, 2, 3, 4, and 5 years. Then create a graph with years on the horizontal axis and FV on the vertical axis to display your results.
- Find the PV of \$1,000 due in 5 years if the discount rate is 10% per year. Again, first work the problem with a formula and then by using the function wizard.
- A security has a cost of \$1,000 and will return \$2,000 after 5 years. What rate of return does the security provide?
- Suppose California's population is 30 million people and its population is expected to grow by 2% per year. How long would it take for the population to double?
- Find the PV of an ordinary annuity that pays \$1,000 at the end of each of the next 5 years if the interest rate is 15%. Then find the FV of that same annuity.
- How would the PV and FV of the above annuity change if it were an annuity due rather than an ordinary annuity?
- What would the FV and PV for parts a and c be if the interest rate were 10% with *semiannual* compounding rather than 10% with *annual* compounding?

- i. Find the PV and FV of an investment that makes the following end-of-year payments. The interest rate is 8%.

Year	Payment
1	\$100
2	200
3	400

- j. Suppose you bought a house and took out a mortgage for \$50,000. The interest rate is 8%, and you must amortize the loan over 10 years with equal end-of-year payments. Set up an amortization schedule that shows the annual payments and the amount of each payment that repays the principal and the amount that constitutes interest expense to the borrower and interest income to the lender.
- (1) Create a graph that shows how the payments are divided between interest and principal repayment over time.
 - (2) Suppose the loan called for 10 years of monthly payments, 120 payments in all, with the same original amount and the same nominal interest rate. What would the amortization schedule show now?

MINI CASE

Assume that you are nearing graduation and have applied for a job with a local bank. The bank's evaluation process requires you to take an examination that covers several financial analysis techniques. The first section of the test addresses discounted cash flow analysis. See how you would do by answering the following questions:

- a. Draw time lines for: (1) a \$100 lump sum cash flow at the end of Year 2, (2) an ordinary annuity of \$100 per year for 3 years, and (3) an uneven cash flow stream of -\$50, \$100, \$75, and \$50 at the end of Years 0 through 3.
- b. (1) What's the *future value* of an initial \$100 after 3 years if it is invested in an account paying 10% annual interest?
 (2) What's the *present value* of \$100 to be received in 3 years if the appropriate interest rate is 10%?
- c. We sometimes need to find out how long it will take a sum of money (or something else, such as earnings, population, or prices) to grow to some specified amount. For example, if a company's sales are growing at a rate of 20% per year, how long will it take sales to double?

- d. If you want an investment to double in 3 years, what interest rate must it earn?
- e. What's the difference between an ordinary annuity and an annuity due? What type of annuity is shown below? How would you change the time line to show the other type of annuity?



- f. (1) What's the future value of a 3-year ordinary annuity of \$100 if the appropriate interest rate is 10%?
 (2) What's the present value of the annuity?
 (3) What would the future and present values be if the annuity were an annuity due?
- g. What is the present value of the following uneven cash flow stream? The appropriate interest rate is 10%, compounded annually.

